DEF 26, 81 – 96 (2003)
DOI: 10.1007/s10203-003-0036-8

Decisions in Economics and Finance

Insuring against the shortfall risk associated with real options

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Received: 7 February 2001 / Accepted: 19 July 2002 – © Springer-Verlag 2003

Abstract. We would like to insure against the risk that a geometric Brownian motion, correlated with the price process of a certain traded asset, is in a set E at time T. In this paper it is shown that the best action one can take to insure against this risk is to buy a binary option on the traded asset. We give explicit formulas in the case that E is an infinite interval. The setting of all our investigations is the Black-Scholes model.

Mathematics Subject Classification (2000): 91B28, 60J65, 62P05, 91B30, 62F03

Journal of Economic Literature Classification: G31

1. Introduction

In this paper we investigate how we can insure against the loss associated with the occurrence of a certain unfortunate event by trading in an asset $\tilde{S}$ and a bond B. The occurrence of the event is assumed to be modelled by the fact that a stochastic process $S_\bullet$ is in a certain set E at time T. We consider the special case that the stochastic process $S_\bullet$ and the price process $\tilde{S}_\bullet$ of the asset $\tilde{S}$ are correlated geometric Brownian motions. The following example illustrates the situation.

I would like to thank G. Pflug for fruitful discussions on drafts of the paper. Further I would like to thank all the people involved in the refereeing process.
This paper was partially written at the Department of Financial and Actuarial Mathematics at the TU-Vienna (research supported in part by SFB 10 of the FWF) and at the Department of Statistics and Decision Support Systems at Vienna University.

Example 1. Suppose that a mining company wants to exploit a certain mining area. In order to do this, the mining company has to make some investments. The mineral M which the mining company would produce after this investment is not traded now, but there exists another mineral $\tilde{M}$ with nearly the same compounds and structure which is already traded on the market. We suppose therefore that the unobservable price process $S_\bullet$ of the mineral M is highly correlated with the price process $\tilde{S}_\bullet$ of the mineral $\tilde{M}$. The mining company faces the following risk: if the price of the mineral M after the period of investment is below the production costs c, then the company will make losses. We will show below that a binary option on $\tilde{M}$ is the best action the company can take if it wants to invest money in options on mineral $\tilde{M}$ to insure against this risk.

From the viewpoint of real options theory (see Amram and Kulatilaka (1999), Dixit and Pindyck (1994) and Trigeorgis (1996)) the investment the companies in the example should make is a real option. Given the price of a real option, one usually asks the question whether one should buy it, i.e., make the investment, or not. Actually, it is not possible to answer this question using "no arbitrage" arguments alone. All other arguments which may be given imply a certain risk. Therefore it is the risk of buying a real option which we shall consider. In particular we show in this article how to deal with such a risk most efficiently.

The risk measure we use to measure the quality of our risk insurance is the shortfall probability. This is the probability that the unfortunate event (the price of the mineral is below c) occurs and our insurance does not pay enough to compensate for the loss. This risk measure and related risk measures are often used when partial hedging is investigated; cf. Föllmer and Leukert (1999, 2000). But in contrast to Föllmer and Leukert (1999) we do not minimize the shortfall probability of a strategy on a stochastic process with respect to a constraint on its equivalent martingale measure, i.e., we do not minimize the shortfall probability of a strategy based on a process observable at any time with respect to this process. Instead, we minimize the shortfall probability of a strategy based on an observable process with respect to another unobservable process. The price for this more general situation is that we can only consider binary options, while Föllmer and Leukert consider arbitrary options. Another important difference with their paper is that our method, as developed so far, only works if the market given by the observable process and the bond is complete, while in Föllmer and Leukert (1999) incomplete markets are also considered.

In this paper we do not use the heavy machinery of stochastic analysis. Instead, we make use of a version of the Neyman–Pearson lemma, some easy lemmas on conditional expectations, and the fact that the Black–Scholes model is complete. In this way we avoid all technical difficulties.

2. The setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let W and $\tilde{W}$ be two standard Brownian motions defined on this space. Suppose that W and $\tilde{W}$ are correlated with fixed correlation $\rho \in (-1, 1) \setminus \{0\}$, i.e., there exists a standard Brownian motion $\tilde{W}^{\perp}$ (also defined on Ω) such that $\tilde{W}^{\perp}$ is independent of $\tilde{W}$ and

$$W = \rho\tilde{W} + \sqrt{1-\rho^2}\tilde{W}^{\perp}.$$

Equivalently $\tilde{W}$ can be expressed as

$$\tilde{W} = \rho W + \sqrt{1-\rho^2}W^{\perp}$$

with $W^{\perp}$ a standard Brownian motion independent of W.

We denote by $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$ the natural (right continuous, saturated) filtration generated by $\tilde{W}$ up to time T. We assume that $W_0 = \tilde{W}_0 = 0$ so that $\mathcal{F}_0$ becomes trivial.

Let S, $\tilde{S}$ be two assets (S not traded and $\tilde{S}$ traded). We denote by $S_\bullet$ and $\tilde{S}_\bullet$ the price processes of S and $\tilde{S}$ and by S_t and $\tilde{S}_t$ these price processes at time t. Let B denote the bond. For simplicity of notation we assume in the following that the interest rate is equal to 0 and that the price processes are given by

$$\frac{dS_t}{S_t} = dW_t,\ \ S_0 = 1 \quad \text{and}$$
$$\frac{d\tilde{S}_t}{\tilde{S}_t} = d\tilde{W}_t,\ \ \tilde{S}_0 = 1.$$

As can be easily verified by the Ito formula (see the book of Steele (2001), Chap. 8) solutions of these equations are given by

$$S_t = e^{W_t - \frac{t}{2}} \quad \text{and } \tilde{S}_t = e^{\tilde{W}_t - \frac{t}{2}}.$$

These solutions are also unique (see Steele (2001), Sect. 9.4).

Thus we obtain the fact that the joint distribution of $\ln(S_T) + \frac{T}{2}$ and $\ln(\tilde{S}_T) + \frac{T}{2}$ is multivariate normal and given by

$$\begin{pmatrix} \ln(S_T) + \frac{T}{2} \\ \ln(\tilde{S}_T) + \frac{T}{2} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} T & \rho T \\ \rho T & T \end{pmatrix}\right). \tag{1}$$

From (1) the conditional distribution of $\ln(\tilde{S}_T) + \frac{T}{2}$ under the condition that $\ln(S_T) + \frac{T}{2} = a$ can be inferred. According to Exercise 3.3 of Steele (2001) we obtain

$$\mathbb{P}\left[\ln(\tilde{S}_T) + \frac{T}{2} \in . \middle| \ln(S_T) + \frac{T}{2} = a\right] \sim N(a\rho, T(1-\rho^2)). \tag{2}$$

We are interested in the following situation.

Suppose we would like to hedge a binary option depending on S_T. We are given the information $(\tilde{S}_{t'})_{t' \in [0,t]}$ and we are allowed to invest in the bond B and in the stock $\tilde{S}$. We are interested in the trading strategies $\phi(t) = (\phi_B(t), \phi_{\tilde{S}}(t))$. Here $\phi_B(t)$ denotes the amount of money held in the bond and $\phi_{\tilde{S}}(t)$ denotes the number of stocks held at time t. We assume that $\phi(t)$ is a predictable process with respect to the filtration $(\tilde{\mathcal{F}}_t)$, that it is self-financing and that at any time $t \in [0, T]$ of non-negative value $V_T^\phi := \phi_{\tilde{S}}(t)\tilde{S}_t + \phi_B(t)$. We further suppose that the initial wealth is given by some fixed value $v_0 = V_0^\phi \in [0, 1]$. We call such strategies admissible and denote the set of all admissible strategies by Φ.

We also introduce the following notations.

We denote the real line by $\mathbb{R}$ and let $\mathbb{R}_+ = [0, \infty)$. By $\mathcal{B}$ (respectively, $\mathcal{B}_+$) we denote the family of Borel sets on $\mathbb{R}$ (respectively, $\mathbb{R}_+$). We let $pr_s : \mathbb{R}^{[0,T]} \mapsto \mathbb{R}$ be the mapping which projects $\mathbb{R}^{[0,T]}$ onto its sth coordinate, i.e., $pr_s((x_t)_{t \in [0,T]}) = x_s$. We denote by $\mathcal{B}_+^{[0,T]}$ the σ-algebra of subsets of $\mathbb{R}_+^{[0,T]}$ generated by $\bigcup_{s \in [0,T]} pr_s^{-1}(\mathcal{B}_+)$.

3. The problem and the main theorems

We consider a set $A_T \in \mathcal{B}_+$, the binary option $H := 1_{A_T}(S_T)$, and the initial value $V_0^\phi := v_0 \in [0, 1]$. We would like to solve the following problem: what are the strategies ϕ such that the shortfall probability, i.e., the probability that the value V_T^ϕ is strictly smaller than H, is minimized within the class of strategies Φ?

To be more precise: find a strategy ϕ which is optimal with respect to the criterion

$$\left.\begin{array}{l} \min\{\mathbb{P}(V_T^\phi < 1_{A_T}(S_T)) : \phi \in \Phi\} \\ \text{subject to the constraint } V_0^\phi = v_0. \end{array}\right\} \tag{3}$$

We will prove the following theorems in Section 4.

Theorem 1. *For $T > 0$, $A_T \in \mathcal{B}_+$, and $v_0 \in [0, 1]$, there exists a set $C_T \in \mathcal{B}_+$ and an admissible strategy $\phi = (\phi_B, \phi_{\tilde{S}})$ which completely replicates $1_{C_T}(\tilde{S}_T)$ and solves (3).*

In some special cases it is possible to obtain the solution of problem (3) in a more specific form.

Theorem 2. *Let F_T be the distribution function of $N(0, T)$, let $\overline{c} = e^{F_T^{-1}(v_0) - \frac{T}{2}}$ and let $\underline{c} = e^{F_T^{-1}(1-v_0) - \frac{T}{2}}$. Then the following relations hold.*

1. *If* $\rho > 0$ *and* A_T *is of the form* $(0, a]$, *then the set* C_T *in Theorem 1 is given by* $C_T = (0, \overline{c}]$ *a.s.*
2. *If* $\rho > 0$ *and* A_T *is of the form* $[a, +\infty)$, *then the set* C_T *in Theorem 1 is given by* $C_T = [\underline{c}, \infty)$ *a.s.*
3. *If* $\rho > 0$ *and* A_T *is bounded and bounded away from* 0, *then the set* C_T *in Theorem 1 is a.s. bounded and a.s. bounded away from* 0.
4. *If* $\rho < 0$ *and* A_T *is of the form* $(0, a]$, *then the set* C_T *in Theorem 1 is given by* $C_T = [\underline{c}, \infty)$ *a.s.*
5. *If* $\rho < 0$ *and* A_T *is of the form* $[a, +\infty)$, *then the set* C_T *in Theorem 1 is given by* $C_T = (0, \overline{c}]$ *a.s.*
6. *If* $\rho < 0$ *and* A_T *is bounded and bounded away from 0, then the set* C_T *in Theorem 1 is a.s. bounded and a.s. bounded away from* 0.

Notice that $\overline{c}$ and $\underline{c}$ are independent of a and $|\rho|$.

Theorem 3. *Let* $\varepsilon, a, T > 0$ *be fixed. Let* ρ *be the correlation of* W *and* $\tilde{W}$, *let* $A_T = (0, a]$, *and let* ϕ *be the solution (given in Theorem 1) of (3) for* $v_0 = \mathbb{E}_{\mathbb{P}}(1_{(0,a]}(S_T)) + \varepsilon$. *Then, as* $\rho \to 1$, *the shortfall probability* $\mathrm{Err}(\rho, \varepsilon, a, T)$, *defined by* $\mathrm{Err}(\rho, \varepsilon, a, T) := \mathbb{P}(V_T^{\phi} < 1_{(0,a]}(S_T))$, *tends to* 0 *faster than any polynomial in* $(1 - \rho)$.

Since $V_T^{\phi} \geq 0$ by the admissibility of ϕ and because of the special structure of H, the constraint optimization problem (3) is equivalent to:

$$\left.\begin{array}{l}\min\{\mathbb{P}(V_T^{\phi} < 1 \mid 1_{A_T}(S_T) = 1) : \phi \in \Phi\} \\ \text{subject to the constraint } V_0^{\phi} = v_0\end{array}\right\},$$

which is further equivalent to

$$\left.\begin{array}{r}\max\{\mathbb{P}(V_T^{\phi} \geq 1 \mid S_T \in A_T) : \phi \in \Phi\} \\ \text{subject to the constraint } V_0^{\phi} = v_0.\end{array}\right\} \tag{4}$$

We denote the wealth at time t along a path ω under the strategy ϕ by $V_t^{\phi}(\omega)$. We let $C_{\phi} \in \mathbb{R}^{[0,T]}$ be the set of all paths ω of $\tilde{S}_{\bullet}$ for which $V_T^{\phi}(\omega) \geq 1$, i.e.,

$$C_{\phi} = \{\omega \in \mathbb{R}^{[0,T]} : \ V_T^{\phi}(\omega) \geq 1\}.$$

Note that, for all ϕ which solve (4),

$$\left.\begin{array}{r}\mathbb{P}(\tilde{S}_{\bullet} \in C_{\phi} \mid S_T \in A_T) = \mathbb{P}(V_T^{\phi} \geq 1 \mid S_T \in A_T) \\ \text{and } \ \mathbb{P}(\tilde{S}_{\bullet} \in C_{\phi}) \leq v_0.\end{array}\right\} \tag{5}$$

We try to solve the more general optimization problem (recall that we assumed $v_0 \in [0, 1]$)

$$\left.\begin{array}{r}\max\{\mathbb{P}(\tilde{S}_{\bullet} \in C \mid S_T \in A_T) : C \in \mathcal{B}_+^{[0,T]}\} \\ \text{subject to the constraint } \mathbb{P}(\tilde{S}_{\bullet} \in C) = v_0.\end{array}\right\} \tag{6}$$

If we can find a set C which solves (6) and for which there exists a ϕ such that $C = C_\phi$, then this ϕ satisfies (5). By (5) and the fact that it satisfies $\mathbb{P}(\tilde{S}_\bullet \in C) = v_0$, this ϕ is also a solution of the optimization problems (4) and (3). We will show in this paper that it is indeed possible to find such a set C and a ϕ with $C = C_\phi$ and thus obtain a solution of (4) and (3).

Problem (6) is now (partially) solved by an application of the following proposition, which can be viewed as a special version of the Neyman–Pearson lemma [see Lehmann (1994)].

Proposition 1. *Let Q and R be measures on a measurable space $(\hat{\Omega}, \mathcal{D})$. Let R be nonatomic and Q absolutely continuous with respect to R. Then for any $v_0 \in (0, 1)$ there exists a unique $\beta \in [0, \infty)$ such that the family $\mathcal{C}$ of all sets $C \in \mathcal{D}$ which solve*

$$\max\{Q(C) : C \in \mathcal{D}\} \text{ subject to the constraint } R(C) = v_0 \tag{7}$$

is nonempty and equals the family of all sets $C \in \mathcal{D}$ which satisfy

$$\begin{aligned} \{\omega \in \hat{\Omega} : \frac{dQ}{dR}(\omega) > \beta\} \subseteq C \subseteq \{\omega \in \hat{\Omega} : \frac{dQ}{dR}(\omega) \geq \beta\} \\ \text{and} \quad R(C) = v_0. \end{aligned} \tag{8}$$

We define probability measures Q^{A_T} and R on $(\mathbb{R}_+^{[0,T]}, \mathcal{B}_+^{[0,T]})$ by

$$Q^{A_T}(D) := \mathbb{P}(\tilde{S}_\bullet \in D \mid S_T \in A_T) \text{ and } R(D) := \mathbb{P}(\tilde{S}_\bullet \in D) \tag{9}$$

for $D \in \mathcal{B}_+^{[0,T]}$. We will show below in the proof of Theorem 1 that R and Q^{A_T} are nonatomic and that Q^{A_T} is absolutely continuous with respect to R.

If we let $\hat{\Omega} = \mathbb{R}_+^{[0,T]}$, $\mathcal{D} = \mathcal{B}_+^{[0,T]}$, and $Q = Q^{A_T}$, then by definition (9) the optimization problems (7) and (6) become identical and thus, by an application of Proposition 1, reformulating (8) in terms of (9) we find that there exists a unique $\beta \in \mathbb{R}_+$ such that a set $C \in \mathcal{B}_+^{[0,T]}$ is a solution of (6) if and only if

$$\begin{aligned} C_{>\beta} &:= \{(x_t)_{t\in[0,T]} \in \mathbb{R}_+^{[0,T]} : \frac{dQ^{A_T}}{dR}((x_t)_{t\in[0,T]}) > \beta\} \subseteq C, \\ C &\subseteq \{(x_t)_{t\in[0,T]} \in \mathbb{R}_+^{[0,T]} : \frac{dQ^{A_T}}{dR}((x_t)_{t\in[0,T]}) \geq \beta\} =: C_{\geq\beta}, \\ &\text{and} \quad R(C) = v_0. \end{aligned} \tag{10}$$

So to solve the problem (6) we only have to calculate $\frac{dQ^{A_T}}{dR}$ and β and then choose C according to condition (10).

In the next section we show that there exists a solution C of our optimization problem (6) which is of the form $C = \{(x_t)_{t\in[0,T]} \;:\; x_T \in C_T\} = pr_T^{-1}(C_T)$ for some set $C_T \in \mathcal{B}_+$.

4. Proofs

4.1. Proof of Theorem 1

For the proof we need some lemmata.

Lemma 1. *Let Q and R be probability measures on $(\hat{\Omega}, \mathcal{D})$ and let $\mathcal{D}_T \subseteq \mathcal{D}$ be a sub σ-algebra. Suppose that*

$$\forall D \in \mathcal{D}, \quad \mathbb{E}_Q(1_D \mid \mathcal{D}_T) = \mathbb{E}_R(1_D \mid \mathcal{D}_T) \; a.s.$$

and that $Q \mid_{\mathcal{D}_T}$ is absolutely continuous with respect to $R \mid_{\mathcal{D}_T}$. Then Q is absolutely continuous with respect to R and

$$\frac{dQ}{dR} = \frac{dQ \mid_{\mathcal{D}_T}}{dR \mid_{\mathcal{D}_T}}.$$

Proof. We have to show that, for all $D \in \mathcal{D}$,

$$Q(D) = \int_D \frac{dQ \mid_{\mathcal{D}_T}}{dR \mid_{\mathcal{D}_T}} dR$$

and do this by the following calculation:

$$\begin{aligned}
Q(D) &= \mathbb{E}_Q(1_D) = \mathbb{E}_{Q|_{\mathcal{D}_T}}(\mathbb{E}_Q(1_D \mid \mathcal{D}_T)) = \mathbb{E}_{Q|_{\mathcal{D}_T}}(\mathbb{E}_R(1_D \mid \mathcal{D}_T)) \\
&= \int \mathbb{E}_R(1_D \mid \mathcal{D}_T) dQ \mid_{\mathcal{D}_T} = \int \mathbb{E}_R(1_D \mid \mathcal{D}_T) \frac{dQ \mid_{\mathcal{D}_T}}{dR \mid_{\mathcal{D}_T}} dR \mid_{\mathcal{D}_T} \\
&= \mathbb{E}_{R|_{\mathcal{D}_T}} \left(\frac{dQ \mid_{\mathcal{D}_T}}{dR \mid_{\mathcal{D}_T}} \mathbb{E}_R(1_D \mid \mathcal{D}_T) \right) \\
&= \mathbb{E}_{R|_{\mathcal{D}_T}} \left(\mathbb{E}_R \left(\frac{dQ \mid_{\mathcal{D}_T}}{dR \mid_{\mathcal{D}_T}} 1_D \mid \mathcal{D}_T \right) \right) \\
&= \mathbb{E}_R \left(\frac{dQ \mid_{\mathcal{D}_T}}{dR \mid_{\mathcal{D}_T}} 1_D \right) = \int_D \frac{dQ \mid_{\mathcal{D}_T}}{dR \mid_{\mathcal{D}_T}} dR.
\end{aligned}$$

Here the third equality follows from the hypothesis and the fifth and seventh equalities follow since $\frac{dQ|_{\mathcal{D}_T}}{dR|_{\mathcal{D}_T}}$ is $\mathcal{D}_T$ measurable. □

Next we define for $A_T \in \mathcal{B}_+$ the measures $Q_T^{A_T}$ and R_T on $(\mathbb{R}_+, \mathcal{B}_+)$ by

$$\left.\begin{aligned}
Q_T^{A_T}(D_T) &:= \mathbb{P}(\tilde{S}_T \in D_T \mid S_T \in A_T) \text{ and} \\
R_T(D_T) &:= \mathbb{P}(\tilde{S}_T \in D_T) \;\text{ with }\; D_T \in \mathcal{B}_+.
\end{aligned}\right\}$$

For ease of notation and calculations we will also consider the measures $\hat{Q}_T^{A_T}$ and $\hat{R}_T$ which are defined on $(\mathbb{R}, \mathcal{B})$ by

$$\hat{Q}_T^{A_T}(D_T) := Q_T^{A_T}\left\{x : \ln(x) + \frac{T}{2} \in D_T\right\}$$

and

$$\hat{R}_T(D_T) := R_T\Big\{x : \ln(x) + \frac{T}{2} \in D_T\Big\}.$$

If we let $\ln(A_T) + \frac{T}{2} := \{\ln(x) + \frac{T}{2} : x \in A_T\}$ we obtain

$$\left.\begin{aligned} \hat{Q}_T^{A_T}(D_T) &= \mathbb{P}(\ln(\tilde{S}_T) + \tfrac{T}{2} \in D_T \mid \ln(S_T) + \tfrac{T}{2} \in \ln(A_T) + \tfrac{T}{2}), \\ \hat{R}_T(D_T) &= \mathbb{P}(\ln(\tilde{S}_T) + \tfrac{T}{2} \in D_T) \;\text{ for }\; D_T \in \mathcal{B}. \end{aligned}\right\} \tag{11}$$

Since by (1) the random variable $\ln(S_T) + \frac{T}{2}$ is N(0,T) distributed and by (2) the conditional distribution of $\ln(\tilde{S}_T) + \frac{T}{2}$ under the condition that $\ln(S_T) + \frac{T}{2} = a$ is $N(a\rho, T(1-\rho^2))$ it is easy to obtain from (11) the fact that the measures $\hat{Q}_T^{A_T}$ and $\hat{R}_T$ are absolutely continuous with respect to Lebesgue measure λ on $(\mathbb{R}, \mathcal{B})$ and their densities are given by

$$\frac{d\hat{Q}_T^{A_T}}{d\lambda} = \frac{\displaystyle\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi T(1-\rho^2)}} e^{-\frac{(x-\rho a)^2}{2(1-\rho^2)T}} \frac{1}{\sqrt{2\pi T}} e^{-\frac{a^2}{2T}} 1_{\ln(A_T)+\frac{T}{2}}(a) d\lambda(a)}{\displaystyle\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{a^2}{2T}} 1_{\ln(A_T)+\frac{T}{2}}(a) d\lambda(a)},$$

$$\frac{d\hat{R}_T}{d\lambda} = \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}.$$

Thus $\hat{Q}_T^{A_T}$ is absolutely continuous with respect to $\hat{R}_T$ and the Radon–Nykodym derivative is given by

$$\frac{d(\hat{Q}_T^{A_T})}{d(\hat{R}_T)}(x) = \frac{\displaystyle\int_{-\infty}^{+\infty} e^{-\frac{(x-\rho a)^2}{2(1-\rho^2)T}} e^{-\frac{a^2}{2T}} 1_{\ln(A_T)+\frac{T}{2}}(a)\, d\lambda(a)}{\sqrt{1-\rho^2} e^{-\frac{x^2}{2T}} \displaystyle\int_{-\infty}^{+\infty} e^{-\frac{a^2}{2T}} 1_{\ln(A_T)+\frac{T}{2}}(a)\, d\lambda(a)}. \tag{12}$$

From this we immediately obtain

Lemma 2. *The measure R_T is nonatomic and the measure $Q_T^{A_T}$ is absolutely continuous with respect to R_T. The Radon–Nykodym derivative is given by*

$$\frac{d(Q_T^{A_T})}{d(R_T)}(x) = \frac{\displaystyle\int_{-\infty}^{+\infty} e^{-\frac{(\ln(x)+\frac{T}{2}-\rho a)^2}{2(1-\rho^2)T}} e^{-\frac{a^2}{2T}} 1_{\ln(A_T)+\frac{T}{2}}(a)\, d\lambda(a)}{\sqrt{1-\rho^2} e^{-\frac{(\ln(x)+\frac{T}{2})^2}{2T}} \int_{-\infty}^{+\infty} e^{-\frac{a^2}{2T}} 1_{\ln(A_T)+\frac{T}{2}}(a)\, d\lambda(a)}.$$

Lemma 3. *Let Q^{A_T} and R be given by (9) and let $\mathcal{B}_T \subset \mathcal{B}_+^{[0,T]}$ be the σ-algebra generated by the projection pr_T, i.e., let*

$$D \in \mathcal{B}_T \;\Leftrightarrow\; D = pr_T^{-1}(D_T) \textit{ for some set } D_T \in \mathcal{B}_+.$$

Then, for all $D \in \mathcal{D}$,

$$E_{Q^{A_T}}(1_D \mid \mathcal{B}_T) = \mathbb{E}_R(1_D \mid \mathcal{B}_T) \ \text{a.s.}$$

Proof. All equalities in this proof are to be considered in the almost sure sense.

$$\begin{aligned}
&Q^{A_T}(D \mid \mathcal{B}_T)((x_t)_{t\in[0,T]}) = \mathbb{P}(\tilde{S}_\bullet \in D \mid S_T \in A_T, \tilde{S}_T = x_T) \\
&= \mathbb{P}\begin{pmatrix} \ln(\tilde{S}_\bullet) + \frac{T}{2} \in \ln(D) + \frac{T}{2} \mid \\ \ln(S_T) + \frac{T}{2} \in \ln(A_T) + \frac{T}{2}, \ln(\tilde{S}_T) + \frac{T}{2} = \ln(x_T) + \frac{T}{2} \end{pmatrix} \\
&= \mathbb{P}\left(\tilde{W} \in \ln(D) + \frac{T}{2} \,\middle|\, W(T) \in \ln(A_T) + \frac{T}{2}, \tilde{W}(T) = \ln(x_T) + \frac{T}{2} \right) \\
&= \mathbb{P}\begin{pmatrix} \tilde{W} \in \ln(D) + \frac{T}{2} \mid \\ \rho\tilde{W}(T) + \sqrt{1-\rho^2}\tilde{W}^\perp \in \ln(A_T) + \frac{T}{2}, \tilde{W}(T) = \ln(x_T) + \frac{T}{2} \end{pmatrix} \\
&= \mathbb{P}\begin{pmatrix} \tilde{W} \in \ln(D) + \frac{T}{2} \mid \\ \rho[\ln(x_T) + \frac{T}{2}] + \sqrt{1-\rho^2}\tilde{W}^\perp \in \ln(A_T) + \frac{T}{2}, \tilde{W}(T) = \ln(x_T) + \frac{T}{2} \end{pmatrix} \\
&= \mathbb{P}\begin{pmatrix} \tilde{W} \in \ln(D) + \frac{T}{2} \mid \\ \tilde{W}^\perp \in \frac{\ln(A_T) + \frac{T}{2} - \rho(\ln(x_T) + \frac{T}{2})}{\sqrt{1-\rho^2}}, \tilde{W}(T) = \ln(x_T) + \frac{T}{2} \end{pmatrix} \\
&= \mathbb{P}\left(\tilde{W} \in \ln(D) + \frac{T}{2} \,\middle|\, \tilde{W}(T) = \ln(x_T) + \frac{T}{2} \right) \\
&= \mathbb{P}\left(\ln(\tilde{S}_\bullet) + \frac{T}{2} \in \ln(D) + \frac{T}{2} \,\middle|\, \ln(\tilde{S}_T) + \frac{T}{2} = \ln(x_T) + \frac{T}{2} \right) \\
&= \mathbb{P}(\tilde{S}_\bullet \in D \mid \tilde{S}_T = x_T) = R(\tilde{S}_\bullet \in D \mid \mathcal{B}_T)((x_t)_{t\in[0,T]}).
\end{aligned}$$

Here the seventh equality holds since $\tilde{W}$ and $\tilde{W}^\perp$ are independent. □

Remark 1. The measures $Q_T^{A_T}$ and R_T are related to Q^{A_T} and R by

$$Q_T^{A_T}(D_T) = Q^{A_T} \mid_{\mathcal{B}_T} (pr_T^{-1}(D_T)) \text{ and } R_T(D_T) = R \mid_{\mathcal{B}_T} (pr_T^{-1}(D_T)) \tag{13}$$

for all $D_T \in \mathcal{B}_T$. Therefore we see from (13) and Lemma 2 that $dQ^{A_T} \mid_{\mathcal{B}_T}$ is absolutely continuous with respect to $dR \mid_{\mathcal{B}_T}$ and thus by (13)

$$\frac{dQ^{A_T} \mid_{\mathcal{B}_T}}{dR \mid_{\mathcal{B}_T}}((x_t)_{t\in[0,T]}) = \frac{dQ_T^{A_T}}{dR_T}(x_T) \text{ a.s.} \tag{14}$$

An application of Lemma 1 and Lemma 3 with $\hat{\Omega} = \mathbb{R}_+^{[0,T]}$, $\mathcal{D} = \mathcal{B}_+$, $\mathcal{D}_T = \mathcal{B}_T$ and $Q = Q^{A_T}$ shows that

$$\frac{dQ^{A_T}}{dR} \text{ exists and } \frac{dQ^{A_T}}{dR}((x_t)_{t\in[0,T]}) = \frac{dQ^{A_T} \mid_{\mathcal{B}_T}}{dR \mid_{\mathcal{B}_T}}((x_t)_{t\in[0,T]}) \text{ a.s.}$$

which together with (14) gives

$$\frac{dQ^{A_T}}{dR}((x_t)_{t\in[0,T]}) = \frac{dQ_T^{A_T}}{dR_T}(x_T) \text{ a.s.} \tag{15}$$

We thus see that there exists a version of $\frac{dQ^{A_T}}{dR}((x_t)_{t\in[0,T]})$ which only depends on the last coordinate x_T of $(x_t)_{t\in[0,T]}$. If we use this version in the definition of $C_{>\beta}$ and $C_{\geq\beta}$, then whether a point $(x_t)_{t\in[0,T]}$ is in the set $C_{>\beta}$ or $C_{\geq\beta}$ defined in (10), depends only on its last coordinate, i.e., it is possible to choose $C_{>\beta}$ and $C_{\geq\beta}$ such that

$$\left.\begin{array}{r} C_{>\beta} = pr_T^{-1}(pr_T(C_{>\beta})) \text{ and } C_{\geq\beta} = pr_T^{-1}(pr_T(C_{\geq\beta})); \\ \text{i.e., } C_{>\beta}, C_{\geq\beta} \in \mathcal{B}_T. \end{array}\right\} \tag{16}$$

Since R_T is nonatomic by Lemma 2 and $R_T(pr_T(D)) = R\mid_{\mathcal{B}_T}(D)$ for all $D \in \mathcal{B}_T$ we see that $R\mid_{\mathcal{B}_T}$ also is nonatomic.

By (16) the sets $C_{>\beta}$ and $C_{\geq\beta}$ are $\mathcal{B}_T$-measurable. We conclude from (10) that

$$R\mid_{\mathcal{B}_T}(C_{>\beta}) \leq v_0 \leq R\mid_{\mathcal{B}_T}(C_{\geq\beta}). \tag{17}$$

Since $R\mid_{\mathcal{B}_T}$ is nonatomic we conclude from (17) that there exists a $\mathcal{B}_T$-measurable set C such that $C_{>\beta} \subseteq C \subseteq C_{\geq\beta}$ and $R\mid_{\mathcal{B}_T}(C) = v_0$. Since $\mathcal{B}_T$-measurable sets are of the form $C = \{((x_t)_{t\in[0,T]}) : x_T \in C_T\}$ we see that there exists a solution of (10) and thus of (6) of the desired form.

We thus obtain the following lemma.

Lemma 4. *The measure R defined by (9) is nonatomic and the measure Q^A defined by (9) is absolutely continuous with respect to R. The Radon–Nykodym derivative $\frac{dQ^{A_T}}{dR} : \mathbb{R}^{[0,T]} \mapsto \mathbb{R}$ can be chosen to depend only on the last coordinate and is given with respect to this coordinate by Lemma 2. The sets $C_{>\beta}$ and $C_{\geq\beta}$ can also be chosen to depend only on their last coordinates, i.e., they can be chosen to be elements of $\mathcal{B}_T^+$. There exists a set $C \in \mathcal{B}_T$ which solves (6), i.e., there exists a solution $C \in \mathbb{R}^{[0,T]}$ of (6) which depends only on the last coordinate. There exists a trading strategy ϕ which replicates $1_C(\tilde{S}_\bullet)$ and this strategy solves (3).*

Proof. That R is nonatomic follows from Lemma 2 and (13). That Q^{A_T} is absolutely continuous with respect to R is a consequence of Lemma 2 and (15). By (15) we also know that $\frac{dQ^{A_T}}{dR} : \mathbb{R}^{[0,T]} \mapsto \mathbb{R}$ can be chosen to depend only on its last coordinate and that, again by (15), it is given with respect to this coordinate by Lemma 2. That the sets $C_{>\beta}$ and $C_{\geq\beta}$ can be chosen to depend only on their last coordinates is the assertion of (16), and that there exists a solution C of (6) which depends only on its last coordinate is the assertion of the paragraph preceding the lemma, which is based on (17).

Thus 1_C is a binary option on $\tilde{S}$ and by completeness of the Black–Scholes model there exists a trading strategy ϕ which completely replicates $1_C(\tilde{S}_\bullet)$, if the initial wealth v_0 equals $\mathbb{P}(\tilde{S}_\bullet \in C)$ as assumed in (6). This trading strategy ϕ therefore solves (3). □

Proof of Theorem 1. Lemma 4 implies Theorem 1 if we let $C_T := pr_T(C)$ with C given by Lemma 4. Then the trading strategy ϕ given by Lemma 4 replicates $1_{C_T}(\tilde{S}_T) = 1_C(\tilde{S}_\bullet)$ and solves (3). □

Remark 2. Theorem 1 remains true in the following more general setting.

(MGS) Let W and $\tilde{W}$ be correlated Brownian motions with correlation $\rho \in (-1, +1) \setminus \{0\}$. Let S and $\tilde{S}$ be assets with price processes given by

$$\frac{dS_t}{S_t} = dW_t + \mu, \ S(0) = 1 \ \text{and}$$
$$\frac{d\tilde{S}_t}{\tilde{S}_t} = d\tilde{W}_t + \tilde{\mu}, \ \tilde{S}(0) = 1$$

and let B be a bond with interest rate $e^{\tilde{\mu}} - 1$.

4.2. *Proofs of Theorems 2 and 3*

Lemma 5. *Let a function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and a measure μ on $(\mathbb{R}, \mathcal{B})$ be given such that f and μ satisfy the following hypotheses:*

(i) *f is strictly positive;*
(ii) *for all $y \in \mathbb{R}$ the function $x \mapsto f(x, y)$ possesses one and only one local extreme point x_y which is a global maximizer and $y < y' \Rightarrow x_y < x_{y'}$;*
(iii) $\lim_{x\to\pm\infty} f(x, y) = 0$ *for all $y \in \mathbb{R}$;*
(iv) *μ is absolutely continuous with respect to Lebesgue measure and*

$$\int_{-\infty}^{+\infty} f(x, y) d\mu(y) = 1.$$

For $A \in \mathcal{B}$ we denote by $f_A : \mathbb{R} \mapsto \mathbb{R}$ the function

$$f_A(x) := \int_A f(x, y) d\mu(y)$$

Then the following conclusions hold.

(I) If $A = (-\infty, \hat{a})$ for some $\hat{a} \in \mathbb{R}$, then f_A is strictly decreasing, $\lim_{x\mapsto-\infty} f_A(x) = 1$ and $\lim_{x\mapsto+\infty} f_A(x) = 0$.
(II) If $A = (\hat{a}, +\infty)$ for some $\hat{a} \in \mathbb{R}$, then f_A is strictly increasing, $\lim_{x\mapsto-\infty} f_A(x) = 0$ and $\lim_{x\mapsto+\infty} f_A(x) = 1$.

(III) If A is a bounded subset of $\mathbb{R}$ and $\mu(A) > 0$, then $f_A(x) > 0$ for any $x \in \mathbb{R}$ and $\lim_{x\to\pm\infty} f_A(x) = 0$.

Remark 3. Item (ii) implies that $f(.\,, y)$ is strictly increasing on $(-\infty, x_y]$ and strictly decreasing on $[x_y, +\infty)$.

Proof. We show first that

$$\begin{array}{c}\text{the measure } \mu \text{ does not vanish on sets}\\ \text{of the form } (-\infty, \hat{a}) \text{ and } (\hat{a}, +\infty).\end{array} \tag{18}$$

We prove this fact by contradiction. Suppose that $\mu(-\infty, \hat{a}) = 0$ for some $\hat{a} \in \mathbb{R}$. Then, for all $x \in \mathbb{R}$,

$$\int_{\hat{a}}^{\infty} f(x, y)d\mu(y) = \int_{-\infty}^{+\infty} f(x, y)d\mu(y) = 1. \tag{19}$$

For $y > \hat{a}$ we see from hypothesis (ii) (see the preceding remark) that $x_y > x_{\hat{a}}$ and thus that $f(.\,, y)$ is strictly increasing on $(-\infty, x_{\hat{a}})$ for $y \geq \hat{a}$. But then we must also have that $x \mapsto \int_{\hat{a}}^{\infty} f(x, y)d\mu(y)$ is strictly increasing on $(-\infty, \hat{a})$ which contradicts (19). So we have proved that μ does not vanish on sets of the form $(-\infty, \hat{a})$. An analogous argument shows that the measure of sets of the form $(\hat{a}, \infty)$ is also strictly positive and thus (18) is proved. □

As another important fact we note that

$$f_{(-\infty,\hat{a}]}(x) = 1 - f_{[\hat{a},\infty)}(x). \tag{20}$$

This fact follows directly from (iv).

We now prove (I).

To prove (I) we show first that, for $\hat{a} \in \mathbb{R}$ fixed, the function

$$x \mapsto f_{(-\infty,\hat{a}]}(x) = \int_{(-\infty,\hat{a}]} f(x, y)d\mu(y)$$

is strictly decreasing on each of the intervals $[x_{\hat{a}}, \infty)$, $(-\infty, x_{\hat{a}}]$. That $f_{(-\infty,\hat{a}]}(x)$ is strictly decreasing on $[x_{\hat{a}}, \infty)$ can be seen as follows.

Since by hypothesis (ii) we have that, for $y < \hat{a}$, the unique local maximizer x_y of $f(.\,, y)$ is smaller than $x_{\hat{a}}$, i.e., $x_y < x_{\hat{a}}$, we see that, for $y \leq \hat{a}$, the functions $f(.\,, y)$ are strictly decreasing on $[x_{\hat{a}}, \infty)$. Therefore and since by (18) the measure μ does not vanish on sets of the form $(-\infty, \hat{a}]$ we see that the function $f_{(-\infty,\hat{a}]} = \int_{-\infty}^{\hat{a}} f(.\,, y)d\mu(y)$ is also strictly decreasing on $[x_{\hat{a}}, \infty)$.

An analogous argument now shows that $f_{[\hat{a},\infty)}(x)$ is strictly increasing on $(-\infty, x_{\hat{a}}]$ and since by (20) we have $f_{(-\infty,\hat{a}]} = 1 - f_{[\hat{a},\infty)}$ we see that $f_{(-\infty,\hat{a}]}$ is also strictly decreasing on $(-\infty, x_{\hat{a}}]$.

We show next that $\lim_{x\to\infty} f_{(-\infty,\hat{a}]}(x) = 0$.

We know by hypothesis (iv) that $f_{(-\infty,\hat{a}]}(x) \leq 1$ and by hypothesis (ii) that

$$\begin{aligned} \cdots x_n > \cdots > x_2 > x_1 > x_{\hat{a}} \text{ and } y \in (-\infty, \hat{a}] \text{ imply that} \\ \cdots > f(x_1, y) > f(x_2, y) > \cdots > f(x_n, y) > \cdots .\} \end{aligned} \tag{21}$$

We now choose an arbitrary strictly increasing sequence $(x_n)_{n\in\mathbb{N}}, x_n \in \mathbb{R}$, with $x_1 > x_{\hat{a}}$ and $\lim_{n\to\infty} x_n = \infty$. Then, by hypothesis (iii) and (21) above, the sequence $(f(x_n, y))_{n\in\mathbb{N}}$ decreases for each $y \in (-\infty, \hat{a}]$ monotonically to 0. Since $\int_{-\infty}^{\hat{a}} f(x_1, y)d\mu(y) \leq 1$ and $f(x_1, y) > f(x_n, y) \geq 0$ we get, by dominated convergence, that the sequence $(f_{(-\infty,\hat{a}]}(x_n) := \int_{-\infty}^{\hat{a}} f(x_n, y)d\mu(y))_{n\in\mathbb{N}}$ tends to 0. Thus we have proved that $\lim_{x\to\infty} f_{(-\infty,\hat{a}]}(x) = 0$.

Analogously we can show that $\lim_{x\to-\infty} f_{(\hat{a},\infty)}(x) = 0$. And by (20) we get from this that $\lim_{x\to-\infty} f_{(-\infty,\hat{a}]}(x) = 1$. Thus (I) has been proved.

Since by (20) we have $f_{[\hat{a},\infty)} = 1 - f_{(-\infty,\hat{a})}$ we obtain (II) from (I).

Finally, we prove (III).

Suppose that A is bounded and $\mu(A) > 0$. Then we have $f_A(x) = \int_A f(x, y)d\mu(y) > 0$ since $f(x, y) > 0$ for all $x, y \in \mathbb{R}$ by hypothesis (i). Let $\hat{a}, b \in \mathbb{R}$ now be such that $A \subseteq [\hat{a}, b]$. Then

$$0 \leq f_A(x) \leq f_{[\hat{a},b]}(x) = 1 - f_{(-\infty,\hat{a})} - f_{(b,\infty)} \tag{22}$$

Since $\lim_{x\mapsto-\infty} f_{(-\infty,\hat{a})} = 1$ and $\lim_{x\mapsto\infty} f_{(b,\infty)} = 1$ we obtain from (22) that $\lim_{x\mapsto\pm\infty} f_A = 0$.

Thus the lemma has been proved. □

Lemma 6. *There exists a version of the conditional probability* $\mathbb{P}(\ln(\tilde{S}_T) + \frac{T}{2} \in D \mid \ln(S_T) + \frac{T}{2} = y) =: \hat{Q}_T^y(D)$ *such that* $\hat{Q}_T^y$ *is absolutely continuous with respect to* $\hat{R}_T$ *and*

$$\frac{d(\hat{Q}_T^y)}{d(\hat{R}_T)} = \frac{e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)T} + \frac{x^2}{2T}}}{\sqrt{1-\rho^2}}.$$

Proof. The result is an immediate consequence of (12) if we consider a proper sequence of sets $(A_T^n)_{n\in\mathbb{N}}$ such that $(\ln(A_T^n) + \frac{T}{2})_{n\in\mathbb{N}}$ converges (with respect to the Hausdorff metric) to y. □

Lemma 7. *Let* $\rho > 0$. *Let* $f : \mathbb{R}^2 \mapsto \mathbb{R}$ *be defined by*

$$f(x, y) = \frac{e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)T} + \frac{x^2}{2T}}}{\sqrt{1-\rho^2}}$$

and let $\mu(y) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} d\lambda(y)$. *Then* f *and* μ *satisfy the hypotheses of Lemma 5.*

Proof. Hypothesis (i) of Lemma 5 follows from the strict positivity of the exponential function. Since we assumed $\rho > 0$ an easy calculation shows that (ii) holds with $x_y = \frac{y}{\rho}$. It is also not difficult to see that (iii) holds if we keep in mind that we assumed $\rho > 0$. So it remains to prove that hypothesis (iv) is fulfilled. That μ is absolutely continuous with respect to Lebesgue measure follows from its definition. To show that $\int_{-\infty}^{+\infty} f(x,y)d\mu(y) = 1$ we note that $\hat{Q}_T^{(-\infty,+\infty)} = \hat{R}_T$ (which follows easily from the definition of Q, R, $\hat{Q}$ and $\hat{R}$). So the integral can be calculated as follows:

$$\int_{-\infty}^{+\infty} f(x,y)d\mu(y) = \frac{\int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)T}+\frac{x^2}{2T}}}{\sqrt{1-\rho^2}} \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} d\lambda(y)}{1}$$
$$= \frac{\int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)T}+\frac{x^2}{2T}}}{\sqrt{1-\rho^2}} \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} d\lambda(y)}{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} d\lambda(y)}$$
$$= \frac{d(\hat{Q}_T^{(-\infty,+\infty)})}{d(\hat{R}_T)} = \frac{d(\hat{R}_T)}{d(\hat{R}_T)} = 1. \qquad \square$$

Proof of Theorem 2. We will only prove the theorem in case 1, i.e., in the case $A_T := (0,a]$. We show first that there exists $\overline{c} \in \mathbb{R}$ such that $C_T := (0,\overline{c}]$ satisfies condition (10). By Proposition 1 and Equation (15) it is sufficient for this to show that $\frac{d(Q_T^{(0,a]})}{d(R_T)}$ is monotonically decreasing, which is clearly equivalent to the fact that $\frac{d(\hat{Q}_T^{(-\infty,\ln(a)+\frac{T}{2}]})}{d(\hat{R}_T)}$ is monotonically decreasing. But this is a consequence of Lemma 5 and Lemma 7 since, if we let $f(.,.)$ and μ be defined as in Lemma 6, then

$$\frac{d(\hat{Q}_T^{(-\infty,\ln(a)+\frac{T}{2}]})}{d(\hat{R}_T)}(x) = \int_{-\infty}^{\ln(a)+\frac{T}{2}} f(x,y)d\mu(y),$$

which is, by conclusion (I) of Lemma 5 (with $\hat{a} := \ln(a) + \frac{T}{2}$), a monotonically decreasing function. So we have proved that C_T is of the form $(0,\overline{c}]$ and thus it remains only to show that $\overline{c} = e^{F_T^{-1}(v_0)-\frac{T}{2}}$. But since, by Lemma 4 and Theorem 1, the set $pr_T^{-1}((0,\overline{c}])$ satisfies (6), we have by (1)

$$F_T(\ln(\overline{c})) + \frac{T}{2} = \mathbb{P}(\tilde{S}_T \in (0,\overline{c}]) = \mathbb{P}(\tilde{S}_\bullet \in pr_T^{-1}((0,\overline{c}])) = v_0. \qquad \square$$

Remark 4. In the more general setting (MGS), introduced before this subsection, instead of Equation (1) the following generalization holds:

$$\begin{pmatrix} \ln(S_T) + \frac{T}{2} - \mu T \\ \ln(\tilde{S}_T) + \frac{T}{2} - \tilde{\mu} T \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} T & \rho T \\ \rho T & T \end{pmatrix} \right) \tag{1'}$$

Using (1′) instead of (1) in all our calculations we obtain the fact that Theorem 2 holds true under (MGS) if we denote by F_T the distribution function of $N(\mu T, T)$ instead of $N(0, T)$.

Proof of Theorem 3. We know by Theorems 1 and 2 that the set C_T which determines our optimal hedge equals a.s. $pr_T^{-1}((0, c(\varepsilon)])$, with $c(\varepsilon)$ a real number depending on ε. Since S_T and $\tilde{S}_T$ possess the same (marginal) distribution, our assumptions imply that

$$R_T((0, c(\varepsilon)]) = v_0 = \mathbb{E}_{\mathbb{P}}(H) + \varepsilon = R_T((0, a]) + \varepsilon. \tag{23}$$

Since R_T is absolutely continuous with respect to Lebesgue measure and $\frac{dR_T}{d\lambda}$ is bounded, we obtain from (23) that $c(\varepsilon) > a + \varepsilon k$ for a suitable constant $k > 0$. The hedging error $\mathrm{Err}(\rho, \varepsilon, a, T)$ is now given by

$$\begin{aligned}
\mathrm{Err}(\rho, \varepsilon, a, T) &= \mathbb{P}\big(V_T^\phi < H\big) = 1 - \mathbb{P}\big(V_T^\phi \geq 1 \big| S_T \in (0, a]\big) \\
&= 1 - \mathbb{P}\big(\tilde{S}_\bullet \in C_\phi \big| S_T \in (0, a]\big) = 1 - \mathbb{P}\big(\tilde{S}_T \in (0, c(\varepsilon)] \big| S_T \in (0, a]\big) \\
&\leq 1 - \mathbb{P}\big(\tilde{S}_T \in (0, a + \varepsilon k] \big| S_T \in (0, a]\big) \\
&= \mathbb{P}\big(\tilde{S}_T \in [a + \varepsilon k, \infty) \big| S_T \in (0, a]\big) \\
&= \mathbb{P}\Bigg(\ln(\tilde{S}_T) + \frac{T}{2} \\
&\qquad \in \left[\ln(a + \varepsilon k) + \frac{T}{2}, \infty\right) \Bigg| \ln(S_T) + \frac{T}{2} \in \left(-\infty, \ln(a) + \frac{T}{2}\right] \Bigg) \\
&= \hat{Q}_T^{(-\infty, \ln(a) + \frac{T}{2}]} \left(\left[\ln(a + \varepsilon k) + \frac{T}{2}, +\infty \right) \right) \\
&\leq \hat{Q}_T^{(-\infty, \ln(a) + \frac{T}{2}]} \left(\left[\ln(a) + \frac{T}{2} + \varepsilon(\ln(a + k) - \ln(a)), \infty \right) \right) \\
&\leq \hat{Q}_T^{\ln(a) + \frac{T}{2}} \left(\left[\ln(a) + \frac{T}{2} + \varepsilon(\ln(a + k) - \ln(a)), \infty \right) \right) \\
&= \int_{\ln(a) + \frac{T}{2} + \varepsilon(\ln(a+k) - \ln(a))}^{+\infty} \frac{d\hat{Q}_T^{\ln(a) + \frac{T}{2}}}{d\hat{R}_T} \frac{d\hat{R}_T}{d\lambda} \, d\lambda
\end{aligned}$$

$$= \int_{\ln(a)+\frac{T}{2}+\varepsilon(\ln(a+k)-\ln(a))}^{+\infty} \frac{e^{-\frac{(x-\rho(\ln(a)+\frac{T}{2}))^2}{2(1-\rho^2)T}}}{\sqrt{1-\rho^2}\sqrt{2\pi T}} d\lambda(x)$$

$$\leq \frac{e^{-\frac{(\ln(a)+\frac{T}{2}+\varepsilon(\ln(a+k)-\ln(a))-\rho(\ln(a)+\frac{T}{2}))^2}{2(1-\rho^2)T}}}{\sqrt{1-\rho^2}\sqrt{2\pi T}} \leq \frac{e^{-\frac{(\varepsilon(\ln(a+k)-\ln(a)))^2}{2(1-\rho^2)T}}}{\sqrt{1-\rho^2}\sqrt{2\pi T}}.$$

Here the third equality follows by Theorem 1, the fourth by Theorem 1 and Theorem 2. The first inequality follows since by (23) $c(\varepsilon) > a + \varepsilon k$. The seventh equality follows from (11), the second inequality follows by concavity of the logarithm and the third inequality by the definition of $\hat{Q}_T^y$ in Lemma 6 and an application of Lemmata 7 and 5. The last equality follows by the definition of $\hat{Q}_T^y$ and the fourth inequality follows easily, for $1-\rho^2$ small, from the fact that, for $y \geq 0$ and α small, $e^{-\frac{y}{\alpha}} \geq \int_y^\infty e^{-\frac{t}{\alpha}} dt$. Finally, since $\varepsilon, k > 0$, an easy calculation shows that, for $\rho \to 1$, the last expression tends faster to 0 than any polynomial in $1-\rho$. □

Remark 5. Again Theorem 3 remains true under (MGS).

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