Isometries with respect to symmetric difference metrics

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§1 INTRODUCTION

Given a measure μ on $\mathbb{E}^d$, we define the *symmetric difference metric* on the space of μ-finite equivalence classes of certain subsets of $\mathbb{E}^d$ by

$$\vartheta_\mu([M]_\mu, [N]_\nu) := \mu(M \triangle N) ,$$

where $[M]_\mu$ denotes the equivalence class containing M. Naturally there arises the question whether *isometries*

$$I : (\mathcal{C}, \vartheta_{\mu_1}) \to (\mathcal{C}, \vartheta_{\mu_2}) ,$$

where $(\mathcal{C}, \vartheta_{\mu_i})$ denotes the space of equivalence classes $[C]_{\mu_i}$, with C convex, are induced by *point-mappings* $\phi : \mathbb{E}^d \to \mathbb{E}^d$ in the sense that

$$I([C]_{\mu_1}) := [\phi(C)]_{\mu_2}$$

holds.

The aim of this paper is to show that *isometries I with respect to symmetric difference metrics* of spaces of convex subsets of $\mathbb{E}^d$ - or more specific *of equivalence classes* of such sets - are induced under rather general conditions *by point mappings* $\phi : A_1 \to A_2$ of certain subsets A_1, A_2 of $\mathbb{E}^d$, and to apply the general result to some important special cases.

The investigation of such problems started with the paper [Gr1] of P. M. Gruber, which is concerned with the case that μ_1 and μ_2 are Lebesgue measure on $\mathbb{E}^d$ and states that under this condition the isometry I is induced by an affinity. It is generalized to *Lebesgue measure on unbounded convex sets* in Theorem 4 of this paper. Theorem 4 is concluded from the first part of the proof of [Gr1] and Theorem 3 which deals essentially with the case of *probability measures* μ_1, μ_2, which are the *restrictions of Lebesgue measure to open connected sets* of $\mathbb{E}^d$. Theorem 3 is a sharp and broad generalization of the paper [Gr2] in which the special case that μ_1 and μ_2 are the restrictions of Lebesgue measure to the unit ball of $\mathbb{E}^d$ is considered. In both theorems we conclude that the *isometry* I is *induced by* a measure preserving *affinity* ϕ. This is caused by the special structure of Lebesgue measure, which is made clear by the proof following Remark 3.

More general than Theorem 3 is Theorem 2 (from which Theorem 3 is an obvious conclusion), which says mainly that given probability measures μ_1 and μ_2, and open connected sets $G_1, G_2 \subset \mathbb{E}^d$ such that

$$\lambda_{|G_i} << \mu_i << \lambda_{|G_i}$$

holds (where λ denotes Lebesgue measure and $<<$ indicates absolute continuity), then *any isometry* I *is induced by a projective mapping* ϕ.

From Theorem 2 we also conclude Theorem 5 and from this Theorem 6. In Theorem 5 and Theorem 6 we consider the case of the invariant measure *on the hyperbolic plane.* The isometries I are in this case *induced by hyperbolic transformations.*

Considering these theorems there arise two questions:

1. How far *may the conditions on the measures* μ_1 and μ_2 *be weakened*, so that the isometry I is still induced by a pointmapping with sufficiently strong geometric and topological properties, so that we may easily conclude by these properties and a form of the fundamental theorem of projective geometry (stated as Lemma 1; for a proof see [Le]) Theorem 2 and so all the other theorems of this paper?

2. Do there exist *essentially smaller subspaces* of $(\mathcal{C}, \vartheta_{\mu_1})$ *such that any isometry* I defined on these subspaces *may be extended* to $(\mathcal{C}, \vartheta_{\mu_1})$?

Both questions are partially answered by the very general Theorem 1 whose proof is for the sake of clearness divided into 6 parts.

In the first step (which is analogous to the beginning of the proof in the paper [Gr2]) we show that half-spaces are mapped onto half-spaces. The *second step* which shows that the *isometry* I is *measure preserving* makes use of the theorem about the invariance of domain and may be seen as the *key for* obtaining the broad *generality* of the result.

By the steps 3 and 4, we see that the *isometry* I can be *extended to* the space of *all* equivalence classes containing *convex sets* (which answers question 2). In the rather lengthy step 5, we prove that I is induced by a measure preserving homeomorphism ϕ which preserves by step 6 the dimension of the affine hull of a set.

So *we obtain under* the rather *natural conditions* of Theorem 1, the result, *that the isometry I may be extended* from the space of (equivalence classes of generalized) simplices to the space of (equivalence classes of) convex sets, *and* that it may be *described by a continuous measure-preserving point mapping $\phi : A_1 \to A_2$, which leaves the dimension of the affine hull of a set invariant.*

Hypotheses $(1), (2), (4)$ and (5) of the theorem cannot be weakened in a reasonable way. In the case of hypothesis (5) this is made clear by Example 1. Whether hypothesis (6) could be weakened is not clear now. *Hypothesis (3) could have been weakened* but this would have led to unnecessary technicalities. That Theorem 1 makes sense in this broad generality is however made clear by Example 2, which together with Corollary 1 indicates that the conclusions of the Theorems 2 and 3 may hold under more general conditions.

The significance of the conditions on A_1 and A_2 in Theorem 1 is made clear by comparison with the work of S. Graf and G. Mägerl [GrMä].

§2 DEFINITIONS AND NOTATIONS

We denote the d-dimensional Euclidean space by $\mathbb{E}^d$, where we always assume $d \geq 2$. We say that a set H is a closed half-space in $\mathbb{E}^d$, if there exists a linear functional l on $\mathbb{E}^d$ such that

$$H = \{x : l(x) \geq r \text{ for some } r \in [-\infty, \infty]\} .$$

(Observe that in the sense of our definition the sets $\mathbb{E}^d$ and $\emptyset$ are also half-spaces.) We call the boundary ∂H of a half-space H a hyperplane. All measures μ (except the hyperbolic measure) we consider are regular measures on the σ-ring of Baire measurable subsets of $\mathbb{E}^d$.

Let μ be a measure. Then we denote by ϑ_μ the symmetric-difference-(pseudo)-metric with respect to μ which is given by $\vartheta_\mu(M, N) := \mu(M \triangle N)$ on the space of μ-finite sets. We denote the space of equivalence classes of μ-finite sets with respect to ϑ_μ by $(\mathcal{M}, \vartheta_\mu)$. If we are given a μ-finite set M we denote the equivalence class containing M by $[M]_\mu$ and call it the μ-equivalence class of M. If $\mathcal{N}$ denotes a family of sets, then we denote by $[\mathcal{N}]$ the family $\{[N] \mid N \in \mathcal{N}\}$. We define $[M]_\mu \cup [N]_\mu := [M \cup N]_\mu$. $[M]_\mu \cup [N]_\mu$ is then well-defined. Analogously we define in general (i.e., also for countably many sets) union, intersection and difference of equivalence-classes and further the symmetric-difference-metric $\vartheta_\mu([M]_\mu, [N]_\mu)$ on the space of equivalence-classes. $[M]_\mu \subseteq [N]_\mu$ means

$[M]_\mu \cap [N]_\mu = [M]_\mu$. (We denote by $\subseteq$ inclusions and by $\subset$ strict inclusions; i.e., $[M]_\mu \subset [N]_\mu$ iff $[M]_\mu \subseteq [N]_\mu$ and $\mu(N \setminus M) > 0$.) If it is clear which measure is meant we write $[.]$ instead of $[.]_\mu$; and if we are given two measures μ_1 and μ_2 we also write $[.]_1$ or $[.]$ instead of $[.]_{\mu_1}$ and $[.]_2$ instead of $[.]_{\mu_2}$.

By $(\mathcal{C}, \vartheta_\mu)$ respectively $(\mathcal{C}_{-\emptyset}, \vartheta_\mu)$ we denote the space of all μ-equivalence-classes containing closed convex sets (as elements), respectively the same without the equivalence-class of the empty set; i.e.

$$(\mathcal{C}, \vartheta_\mu) := (\mathcal{M}(X), \vartheta_\mu) \cap \{[C] : C \text{ closed convex }\}$$

and

$$(\mathcal{C}_{-\emptyset}, \vartheta_\mu) := (\mathcal{C}, \vartheta_\mu) \setminus [\emptyset] \ .$$

Analogously we define $(\mathcal{H}, \vartheta_\mu)$, $(\mathcal{H}_{-\emptyset}, \vartheta_\mu)$, $(\mathcal{P}, \vartheta_\mu)$, $(\mathcal{P}_{-\emptyset}, \vartheta_\mu)$, $(\mathcal{Q}, \vartheta_\mu)$, $(\mathcal{Q}_{-\emptyset}, \vartheta_\mu)$, where H stands for closed half-space, P stands for polyhedron and Q for generalized simplex. (By a polyhedron we mean a d-dimensional convex set, possibly unbounded, that is the intersection of a finite family of closed halfspaces. By a generalized simplex we mean a d-dimensional convex set which is either a simplex or which can be obtained from a simplex by shifting some but not all of its vertices to infinity; i.e., a set affine equivalent to a set $\{(x_1, ..., x_d) \in \mathbb{E}^d \mid x_1, ..., x_d \geq 0, x_1 + ... + x_k \leq 1\}$, where $0 \leq k \leq d$.)

Let further

$$(\mathcal{R}, \vartheta_\mu) := \{[Q_1] \cap [Q_2] \mid [Q_1], [Q_2] \in (\mathcal{Q}, \vartheta_\mu)\} \, .$$

Given a measure μ and a set $Y \subset \mathbb{E}^d$, we denote by $tr(Y, \mu)$ the support of the restriction of μ to Y. By $conv(M)$ we denote the convex hull of a set M. If we are given a family of sets $\mathcal{M}$ we denote by $conv(\mathcal{M})$ the family $\{conv(M) \mid M \in \mathcal{M}\}$. If μ is absolutely continuous with respect to ν we write $\mu << \nu$; the Lebesgue measure is denoted by λ. If nothing else is stated i denotes an element of $\{1, 2\}$. By an isometry between metric spaces we mean an isometry into.

§3 THE MAIN THEOREM

We will often make use of the following easily proved proposition.

Proposition 1

Let μ_1 and μ_2 be measures on a space X, let $(\mathcal{D}_i, \vartheta_{\mu_i}) \subseteq (\mathcal{M}, \vartheta_{\mu_i})$ for $i \in \{1, 2\}$, and let

$$I : (\mathcal{D}_1, \vartheta_{\mu_1}) \to (\mathcal{D}_2, \vartheta_{\mu_2})$$

be a measure preserving isometry (i.e. $\mu_2(I[D_1]) = \mu_1(D_1)$ for $[D_1] \in \mathcal{D}_1$). Then I and I^{-1} preserve inclusions as well as strict inclusions.$\Box$

The following form of the fundamental theorem of projective geometry is easily obtained from [Le].

Lemma 1 (Fundamental Theorem of Projective Geometry)

Let $d \geq 2$, G an open connected subset of P^d and $\phi' : G \to P^d$ a continuous injective mapping which maps points of any segment contained in G into points contained in a line. Then there exists a projective isomorphism $f : P^d \to P^d$ which coincides with ϕ' on G.

From this we get immediately:

Lemma 2 (Fundamental Theorem of Affine Geometry)

Let $d \geq 2$ and $\phi : \mathbb{E}^d \to \mathbb{E}^d$ be a continuous injective mapping which maps points of a line into points contained in a line, then ϕ is an affinity.$\Box$

We are now able to prove the main theorem of this paper. In what follows, we topologize the space of half spaces $\mathcal{H}$ in the following way. At $H_{u,t} = \{x \in \mathbb{E}^d \mid \langle x, u \rangle \leq t\}$ ($u \in S^{d-1}$, $t \in \mathbb{R}$) a neighbourhood base is given by $\{H_{u',t'} \mid u' \in S^{d-1}, t' \in \mathbb{R}, \|u' - u\| < \varepsilon, |t' - t| < \varepsilon\}$, and at $H = \emptyset$ and $H = \mathbb{E}^d$ a neighbourhood base is given by $\{\emptyset\} \cup \{H_{u,t} \mid u \in S^{d-1}, t \in \mathbb{R}, t < -1/\varepsilon\}$ and $\{\mathbb{E}^d\} \cup \{H_{u,t} \mid u \in S^{d-1}, t \in \mathbb{R}, t > 1/\varepsilon\}$, respectively.

Theorem 1

Let μ_1, μ_2 be two measures on $\mathbb{E}^d$ and let $(\mathcal{D}_1, \vartheta_{\mu_1})$ and $(\mathcal{D}_2, \vartheta_{\mu_2})$ be spaces of equivalence classes of μ_1- respectively μ_2-finite sets such that the following hypotheses are fulfilled:

(1) μ_1 and μ_2 are probability measures,
(2) μ_1 and μ_2 vanish on each hyperplane,
(3) given two different closed half-spaces H and H'

$$0 < \mu_i(H) < 1 \Rightarrow \mu_i(H \triangle H') > 0$$

holds,
(4) the inclusions

$$(\mathcal{H}_{-\emptyset}, \vartheta_{\mu_1}) \subseteq (\mathcal{D}_1, \vartheta_{\mu_1}) \subseteq (\mathcal{C}, \vartheta_{\mu_1})$$

and

$$(\mathcal{D}_2, \vartheta_{\mu_2}) \subseteq (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_2})$$

are fulfilled,
(5) for $[D]_2, [D']_2 \in (\mathcal{D}_2, \vartheta_{\mu_2})$ *with* $[D]_2 = [\mathbb{E}^d]_2 \setminus [D']_2$ *there exists a closed half-space* H *such that* $H \in [D]_2$.

Then any isometry
$$I : (\mathcal{D}_1, \vartheta_{\mu_1}) \to (\mathcal{D}_2, \vartheta_{\mu_2})$$
is a measure preserving mapping which maps $(\mathcal{H}_{1-\emptyset}, \vartheta_{\mu_1})$ *homeomorphically onto* $(\mathcal{H}_{2-\emptyset}, \vartheta_{\mu_2})$.

(6) If in addition
$$(\mathcal{Q}_{-\emptyset}, \vartheta_{\mu_1}) \subseteq (\mathcal{D}_1, \vartheta_{\mu_1})$$
holds then there exists a unique measure preserving, bijective, isometric extension
$$\widetilde{I} : (\mathcal{C}, \vartheta_{\mu_1}) \to (\mathcal{C}, \vartheta_{\mu_2})$$
of I *which maps* $(\mathcal{P}, \vartheta_{\mu_1})$ *onto* $(\mathcal{P}, \vartheta_{\mu_2})$.

Further there exist sets A_i $(i \in \{1, 2\})$ *such that* $\mu_i(A_i) = 1$ *and*

$$A_i \supseteq int(conv(tr(\mathbb{E}^d, \mu_i))) \cap tr(\mathbb{E}^d, \mu_i)$$

and there exists a homeomorphism ϕ *from* A_1 *onto* A_2 *inducing the isometry* I *and also the isometry* $\widetilde{I}$*; i.e. such that*

$$M \subset A_1 \text{ and } [M] \in (\mathcal{C}, \vartheta_{\mu_1})$$

implies
$$[\phi(M)]_2 = \widetilde{I}([M]).$$
Further ϕ *as well as* ϕ^{-1} *are measure preserving, and a set* $M \subset A_1$ *is affinely independent if and only if* $\phi(M)$ *is affinely independent (which implies that the dimension of the affine hull of a set is invariant under* ϕ *as well as* ϕ^{-1}*).*

If the sets $tr(\mathbb{E}^d, \mu_i)$ $i \in \{1, 2\}$ *are bounded we may choose* A_1 *and* A_2 *by* $A_i := tr(\mathbb{E}^d, \mu_i)$*. (So in this case the mapping* ϕ *becomes a homeomorphism from* $tr(\mathbb{E}^d, \mu_1)$ *onto* $tr(\mathbb{E}^d, \mu_2)$*.)*

If $tr(\mathbb{E}^d, \mu_1) = \mathbb{E}^d$ *or* $tr(\mathbb{E}^d, \mu_2) = \mathbb{E}^d$ *then* ϕ *becomes an affinity.*

Remark 1

Of course the theorem also holds if we replace hypothesis (1) by the hypothesis $0 < \mu_1(\mathbb{E}^d) = \mu_2(\mathbb{E}^d) < \infty$.

Hypothesis (2) can also be formulated in the following way:

The measures μ_i, $i \in \{1,2\}$ vary continuously on the space of (closed) half-spaces. If we refer to hypothesis (2) in the following we mean one of the two equivalent versions.

During the proof (end of the secod part) it turns out that $[\emptyset] \notin (\mathcal{D}_1, \vartheta_{\mu_1})$ so that we have $(\mathcal{H}_{-\emptyset}, \vartheta_{\mu_1}) \subseteq (\mathcal{D}_1, \vartheta_{\mu_1}) \subseteq (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_1})$. But it is enough to assume the weaker hypothesis (4) to prove the theorem.

We also mention that the spaces $(\mathcal{C}_i, \vartheta_{\mu_i})$ and $(\mathcal{D}_i, \vartheta_{\mu_i})$ are in general not complete.

Proof of Theorem 1: The proof will be divided into 6 parts. Because of (2) we may consider in the first two parts arbitrary half-spaces instead of closed ones. In the following i will, if not otherwise specified, always denote an element of the set $\{1,2\}$.

1^{st} *Part:* *Equivalence classes which contain a half-space are mapped by I onto equivalence classes which also contain a half-space (see also [Gr2]).*

We show that the space of equivalence classes $(\mathcal{H}, \vartheta_{\mu_1})$ which contain a half-space is mapped by I^* into $(\mathcal{H}, \vartheta_{\mu_2})$. Here I^* denotes the mapping, which is defined on $(\mathcal{H}, \vartheta_{\mu_1})$, coincides on $\mathcal{H}_{-\emptyset}$ with I and maps $[\emptyset]$ onto $[\mathbb{E}^d]_2 \setminus I([\mathbb{E}^d])_2$. Because for $[H] \in (\mathcal{H}, \vartheta_{\mu_2})$

$$\vartheta_{\mu_2}(I^*([H]), I^*([\emptyset])) = \vartheta_{\mu_2}(I([H]), [\mathbb{E}^d]_2 \setminus I([\mathbb{E}^d])) =$$

$$= 1 - \vartheta_{\mu_2}(I([H])), I([\mathbb{E}^d])) = 1 - \vartheta_{\mu_1}([H], [\mathbb{E}^d]) = \vartheta_{\mu_1}([H], [\emptyset])$$

holds, I^* is also an isometry and therefore injective.(If $[\emptyset]$ were in $(\mathcal{D}_1, \vartheta_{\mu_1})$ then I and I^* would coincide on the whole space $(\mathcal{H}, \vartheta_{\mu_1})$, since I is an isometry. But it will be shown at the end of the second part of our proof that only $[\emptyset] \notin (\mathcal{D}_1, \vartheta_{\mu_1})$ can hold, since otherwise we would have $I([\emptyset]) = I^*([\emptyset]) = [\emptyset]_2$, which contradicts $[\emptyset]_2 \notin (\mathcal{D}_2, \vartheta_{\mu_2})$ and thus hypothesis (5).)

By the following calculation we obtain that the image of an equivalence class of a half-space H, $[H] \notin \{[\emptyset], [\mathbb{E}^d]\}$ under I^* is again an equivalence class of a half-space:

$$1 = \vartheta_{\mu_1}([H], [\mathbb{E}^d \setminus H]) = \vartheta_{\mu_2}(I^*([H]), I^*([\mathbb{E}^d \setminus H])) =$$

$$= \mu_2(I^*([H]) \cup I^*([\mathbb{E}^d \setminus H])) - \mu_2((I^*([H]) \cap I^*([\mathbb{E}^d \setminus H])) \leq$$

$$\leq \mu_2(I^*([H]) \cup I^*([\mathbb{E}^d \setminus [H])) \leq 1;$$

therefore

$$\mu_2(I^*([H]) \cup I^*([\mathbb{E}^d \setminus H])) = 1 \text{ and } \mu_2(I^*([H]) \cap I^*([\mathbb{E}^d \setminus H])) = 0;$$

and finally

$$I^*([H]) = [\mathbb{E}^d]_2 \setminus I^*([\mathbb{E}^d \setminus H]).$$

Because $I^*([H]), I^*([\mathbb{E}^d \setminus H]) \in (\mathcal{D}_2, \vartheta_{\mu_2})$ there exists by (5) a closed half-space G with $[G]_2 = I^*([H])$.

If $[H] = [\mathbb{E}^d]$ or $[H] = [\emptyset]$ we consider a sequence $\langle [H_n] \rangle_{n\in\mathbf{N}}$ of equivalence classes of half-spaces converging to $[H]$ with $0 < \mu_1([H_n]) < 1$ and $\vartheta_{\mu_1}([H_n],[H_{n+1}]) < \frac{1}{2^n}$. (The existence of such a sequence is granted by (2).) The Cauchy sequences $\langle I^*([H_n]) \rangle_{n\in\mathbf{N}}$ and $\langle I^*([\mathbb{E}^d] \setminus [H_n]) \rangle_{n\in\mathbf{N}}$ converge towards

$$I^*([H]) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} I^*([H_n])$$

and

$$I^*([\mathbb{E}^d] \setminus [H]) = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} I^*([\mathbb{E}^d] \setminus [H_n])\,.$$

But by the argument given before (using $[H_n] \notin \{[\emptyset], [\mathbb{E}^d]\}$) we get

$$I^*([H_n]) = [\mathbb{E}^d]_2 \setminus I^*([\mathbb{E}^d] \setminus [H_n])$$

and now an application of the duality relations (of intersection and union) yields

$$I^*([H]) = [\mathbb{E}^d]_2 \setminus I^*([\mathbb{E}^d] \setminus [H])\,. \tag{1.1}$$

Together with (5) we get again a closed half-space G with $[G]_2 = I^*([H])$. We obtain now that the I^*-image of an arbitrary element $(\mathcal{H}, \vartheta_{\mu_1})$ must be in $(\mathcal{H}, \vartheta_{\mu_2})$ and that for arbitrary $[H] \in (\mathcal{H}, \vartheta_{\mu_1})$ equation (1.1) is fulfilled.

<u>2^{nd} *Part:*</u> *I is measure- and inclusion-preserving.*

We show first that $(\mathcal{H}, \vartheta_{\mu_i})$ is homeomorphic to S^d.

Therefore we define mappings

$$f'_i : S^{d-1} \times [0,1] \mapsto (\mathcal{H}, \vartheta_{\mu_i})$$

by

$$f'_i(x,\xi) = [H^i_{(x,\xi)}]_i\,,$$

where $H^i_{(x,\xi)}$ denotes a half-space H, which fulfills $\mu_i([H]) = \xi$ and $H = \{y \mid \langle x, y\rangle \geq \beta\}$ for a suitable $\beta \in [-\infty, \infty]$. Since the measures μ_i are by (1) probability measures and vary by (2) continuously on half-spaces, the mappings f_i' are well-defined and surjective.

Further, if we denote by q the quotient mapping from $S^{d-1} \times [0,1]$ onto $S^{d-1} \times [0,1]/S^{d-1} \times \{0,1\}$ there exist well-defined mappings

$$f_i : S^{d-1} \times [0,1]/S^{d-1} \times \{0,1\} \to (\mathcal{H}, \vartheta_{\mu_i})$$

such that $f_i' = f_i \circ q$. By (3) these mappings are injective and thus bijective. (We show that the functions f_i' separate different points $(x,\xi), (y,\eta) \in S^{d-1} \times (0,1)$. If $\xi \neq \eta$ this is clear. If $x \neq y$ this follows, because $H^i_{(x,\xi)} \neq H^i_{(y,\eta)}$ and $0 < \mu_i([H^i_{(x,\xi)}]) = \xi < 1$ by (3).) To show that the mappings f_i map the quotient space $S^{d-1} \times [0,1]/S^{d-1} \times \{0,1\}$ homeomorphic onto $(\mathcal{H}, \vartheta_{\mu_i})$, it is enough, by compactness of the quotient space and because $f_i' = f_i \circ q$ and q is a quotient mapping, to deduce continuity of the mappings f_i'.

Consequently we show that given an arbitrary point $(x,\xi) \in S^{d-1} \times [0,1]$ and an arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $(y,\eta) \in S^{d-1} \times [0,1]$ with

(2.1) $$\|x - y\| < \delta \text{ and } |\xi - \eta| < \delta$$

we have

$$\vartheta_{\mu_i}([H^i_{(x,\xi)}], [H^i_{(y,\eta)}]) < 2\varepsilon\,.$$

So let (x,ξ) and $1 > \varepsilon > 0$ be given. Let β be such that $H^i_{(x,\xi)} = \{z \mid \langle x, z\rangle \geq \beta\}$. Then we choose $n \in \mathbf{N}$ such that

(2.2) $$\mu_i(nB^d) > 1 - \frac{\varepsilon}{4}$$

and, if $|\beta| < \infty$, such that $|\beta| < n$. Further we choose δ such that

(2.3) $$\delta < \frac{\varepsilon}{2}\,,$$

(2.4) $$\mu_i(\{z \in nB^d \mid \beta - 2n\delta \leq \langle x, z\rangle \leq \beta + 2n\delta\}) < \frac{\varepsilon}{2}$$

and, if $|\beta| < \infty$ also

(2.5) $$|\beta| + 2n\delta < n$$

holds. That δ may be chosen to fulfil condition (2.4) is a consequence of hypothesis (2), which says that the measures μ_i vary continuously on half-spaces.

Further we notice that β may also equal $+\infty$ or $-\infty$. Let now (y,η) be such that $\|x-y\|<\delta$ and $|\xi-\eta|<\delta$.

We distinguish two cases:

1^{st} One of the two sets $C_1 := H^i_{(x,\xi)} \cap nB^d$ and $C_2 := H^i_{(y,\eta)} \cap nB^d$ contains the other.

2^{nd} The intersection $\partial H^i_{(x,\xi)} \cap \partial H^i_{(y,\eta)} \cap nB^d$ is not empty.

In the first case (and if $\beta \in \{-\infty,+\infty\}$ only this case can occur) we assume w.l.o.g. that $C_2 \subseteq C_1$. Because of (2.1), (2.2) and (2.3) we get

$$\begin{gathered}\vartheta_{\mu_i}([H^i_{(x,\xi)}]_i,[H^i_{(y,\eta)}]_i) < 2\mu_i(\mathbb{E}^d \setminus nB^d) + \mu_i(C_1 \setminus C_2) \leq \\ \leq 2\frac{\varepsilon}{4} + |\mu_i(C_1) - \mu_i(C_2)| = \\ = \frac{\varepsilon}{2} + |\mu_i(C_1) - \xi| + |\xi-\eta| + |\eta - \mu_i(C_2)| \leq \\ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \delta + \frac{\varepsilon}{4} < \frac{3\varepsilon}{2}\,.\end{gathered}$$

In the second case we obtain that

$$|\langle x,z\rangle - \beta| \leq 2n\delta$$

if $z \in \partial H^i_{(y,\eta)} \cap nB^d$. To see this let $\gamma := \langle y,z\rangle$ and let w be an arbitrary point of intersection of

$$\partial H^i_{(x,\xi)} = \{w \mid \langle x,w\rangle = \beta\}$$

and

$$\partial H^i_{(y,\eta)} = \{w \mid \langle y,w\rangle = \gamma\}\,,$$

in nB^d. Then we get, since $\|z\|,\|w\| \leq n$ and because of (2.1) $\|x-y\|<\delta$,

$$\begin{gathered}|\langle x,z\rangle - \beta| \leq |\langle x,z\rangle - \langle y,z\rangle| + |\langle y,z\rangle - \beta| \leq \\ \leq \|x-y\|\|z\| + |\gamma-\beta| \leq n\delta + |\langle y,w\rangle - \langle x,w\rangle| \leq \\ n\delta + \|x-y\|\|w\| \leq 2n\delta\,.\end{gathered}$$

This can also be written in the form

$$\beta - 2n\delta \leq \langle x,z\rangle \leq \beta + 2n\delta$$

which implies immediately

$$\partial H^i_{(y,\eta)} \cap nB^d \subseteq \{z \mid \beta - 2n\delta \leq \langle x,z\rangle \leq \beta + 2n\delta\} \cap nB^d\,.$$

If we set

$$D := (H^i_{(x,\xi)} \triangle H^i_{(y,\eta)}) \cap nB^d$$

we find that

$$D \subset \{z \mid \beta - 2n\delta \leq \langle x, z\rangle \leq \beta + 2n\delta\} \cap nB^d \tag{2.6}$$

or

$$nB^d \setminus D \subset \{z \mid \beta - 2n\delta \leq \langle x, z\rangle \leq \beta + 2n\delta\} \cap nB^d. \tag{2.7}$$

If (2.6) holds we obtain with (2.2) and (2.4)

$$\vartheta_{\mu_i}([H^i_{(x,\xi)}], [H^i_{(y,\eta)}]) \leq 2\mu_i(\mathbb{E}^d \setminus nB^d) + \vartheta_{\mu_i}([H^i_{(x,\xi)} \cap nB^d], [H^i_{(y,\eta)} \cap nB^d]) \leq$$

$$\frac{\varepsilon}{2} + \mu_i(\{z \mid \beta - 2n\delta \leq \langle x, z\rangle \leq \beta + 2n\delta\} \cap nB^d) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

However (2.7) cannot hold, since $|\beta| < \infty$, and thus (2.5) and (2.7) would imply $\|x-y\| \geq \sqrt{2}$, which contradicts $\|x-y\| < \delta$. The inequality $\|x-y\| \geq \sqrt{2}$ can be seen as follows:
Because of (2.5)

$$nB^d \setminus \{z \mid \beta - 2n\delta \leq \langle x, z\rangle \leq \beta + 2n\delta\}$$

has two connected components K_1 and K_2 with $nx \in K_1$ and $-nx \in K_2$. By the definition of $H^i_{(x,\xi)}$ we get

$$nx \in H^i_{(x,\xi)} \quad \text{and} \quad -nx \notin H^i_{(x,\xi)}$$

and, since (2.7) would imply

$$K_1 \cup K_2 \subset D\,,$$

we would obtain finally

$$-nx \in H^i_{(y,\eta)} \quad \text{and} \quad nx \notin H^i_{(y,\eta)}\,, \tag{2.8}$$

which would imply that the angle enclosed by x and y is greater than $\frac{\pi}{2}$; i.e. $|x-y| \geq \sqrt{2}$. Contradiction.

So we have proved the continuity of the mappings f_i', and thus also that the functions f_i are homeomorphisms.

The mapping

$$h : S^{d-1} \times [0,1]/S^{d-1} \times \{0,1\} \to S^{d-1} \times [0,1]/S^{d-1} \times \{0,1\}\,,$$

which is defined by

$$h := f_2^{-1} \circ I^* \circ f_1$$

is injective and continuous and thus, since $S^{d-1} \times [0,1]/S^{d-1} \times \{0,1\}$ is homeomorphic to S^d, it is also surjective. (This follows from the theorem on the invariance of domain, since S^d is the only non-empty open compact subset of S^d (see [HuWa]).) Thus

$$I^* = f_2 \circ h \circ f_1^{-1}$$

is also surjective.

Since $[\emptyset]_2 \notin Im(I)$, we infer from hypothesis (4) and the definition of I^* that $I^*([H]) = [\emptyset]_2$ if and only if $[H] = [\emptyset]$. So

$$I^*([\emptyset]) = [\emptyset]_2 \ , \tag{2.9}$$

and

$$I : (\mathcal{H}_{1-\emptyset}, \vartheta_{\mu_1}) \mapsto (\mathcal{H}_{2-\emptyset}, \vartheta_{\mu_2}) \tag{2.10}$$

is a homeomorphism and $[\emptyset] \notin (\mathcal{D}_1, \vartheta_{\mu_1})$. Thus considering the equivalence classes of the complements we get with (2.9) and (1.1)

$$I([\mathbb{E}^d]) = [\mathbb{E}^d]_2 \ . \tag{2.11}$$

We denote now the extension of I, which equals I on $(\mathcal{D}_1, \vartheta_{\mu_1})$ and maps $[\emptyset]$ to $[\emptyset]$, again by I. Since (2.11) implies

$$\mu_1([D]) = 1 - \vartheta_{\mu_1}([\mathbb{E}^d], [D]) = 1 - \vartheta_{\mu_2}(I([\mathbb{E}^d]), I([D])) = \mu_2(I([D])) \, ,$$

I is also a measure preserving isometry of $(\mathcal{D}_1, \vartheta_{\mu_1}) \cup \{[\emptyset]\}$ onto its image. So I^{-1} is also measure preserving, and thus Proposition 1 implies that I and I^{-1} preserve inclusions as well as strict inclusions.

<u>3^{rd} *Part:*</u> *We extend the isometry I to the space of equivalence classes of polyhedra $(\mathcal{P}, \vartheta_{\mu_1})$.*

Let P be a polyhedron given by

$$[P] = \bigcap [\mathcal{H}] \ ,$$

where $[\mathcal{H}]$ denotes a finite family of equivalence classes of half-spaces. Now we will show that

$$\mu_1([P]) = \mu_2(\bigcap I([\mathcal{H}])) \ : \tag{3.1}$$

First we consider polyhedra $P = \bigcap \mathcal{H}$ for which $I([P])$ exists. These contain by condition (6) all generalized simplices. Since I is inclusion preserving we have $I([P]) \subseteq \bigcap I([\mathcal{H}]))$. Since I^{-1} also preserves inclusions as well as strict inclusions, $I([P]) \neq \bigcap I([\mathcal{H}])$ would imply

$$[P] = I^{-1}(I([P]) \subset I^{-1}(\bigcap I([\mathcal{H}]))$$

and

$$I^{-1}(\bigcap I([\mathcal{H}])) \subseteq \bigcap I^{-1}(I([\mathcal{H}])) = \bigcap [\mathcal{H}] = [P]$$

and thus

$$[P] \neq [P] ,$$

which is a contradiction.

Thus we conclude:

(3.1*) Given a measure preserving isometry I, a finite set of half-spaces $\mathcal{H}$ and an equivalence class $[P]$ of a polyhedron P, on which I is defined, we have

$$[P] = \bigcap [\mathcal{H}] \;\Rightarrow\; I([P]) = \bigcap I([\mathcal{H}])$$

and, since I is measure preserving,

$$[P] = \bigcap [\mathcal{H}] \;\Rightarrow\; \mu_1([P]) = \mu_2(\bigcap I([\mathcal{H}]))$$

holds.

In particular we have shown

(3.1') $$\bigcap [\mathcal{H}] = [\emptyset] \;\Rightarrow\; \bigcap I([\mathcal{H}]) = [\emptyset]_2.$$

Let now P be an arbitrary polyhedron, let

(3.2) $$[P] = \bigcap [\mathcal{H}] ,$$

where $\mathcal{H}$ denotes a finite set of half-spaces and let

(3.3) $$[P] = \bigcup [\mathcal{S}] ,$$

where $\mathcal{S}$ denotes a finite family of generalized simplices with pairwise disjoint interiors.

Let S_1, S_2 be different elements of $\mathcal{S}$, then we have $[S_1] \cap [S_2] = [\emptyset]$, since by (2) μ_1 vanishes on hyperplanes. Furthermore $I([S_1]) \cap I([S_2]) = [\emptyset]$, since I is a measure preserving isometry. Thus (3.3) implies

$$\mu_1(P) = \mu_2(\bigcup I([\mathcal{S}])) . \tag{3.4}$$

Since I preserves inclusions, we get from (3.2) and (3.3)

$$\bigcup I([\mathcal{S}]) \subseteq \bigcap I([\mathcal{H}]) . \tag{3.5}$$

We show next

$$\bigcup I([\mathcal{S}]) \supseteq \bigcap I([\mathcal{H}]) . \tag{3.6}$$

Indirectly, suppose $\bigcup I([\mathcal{S}]) \not\supseteq \bigcap I([\mathcal{H}])$. Then there exists a finite family of half-spaces $\mathcal{G}$ for which

$$\bigcap [\mathcal{G}]_2 \cap \bigcap I([\mathcal{H}]) \neq [\emptyset]_2 \tag{3.7}$$

as well as

$$\bigcap [\mathcal{G}]_2 \cap \bigcup I([\mathcal{S}]) = [\emptyset]_2 \tag{3.8}$$

hold. From (3.7) and (3.1') we conclude

$$\bigcap I^{-1}([\mathcal{G}]_2) \cap \bigcap [\mathcal{H}] \neq [\emptyset] ,$$

which together with (3.2) and (3.3) implies that there exists $S \in \mathcal{S}$, such that

$$\bigcap I^{-1}([\mathcal{G}]_2) \cap [S] \neq [\emptyset] .$$

Therefore there exists a simplex T with $\mu_1([T]) > 0$ and

$$[T] \subset \bigcap I^{-1}([\mathcal{G}]_2) \cap [S] .$$

Since I is inclusion preserving, we obtain

$$I([T]) \subset \bigcap [\mathcal{G}]_2 \cap I([S]) ,$$

and together with (3.8)

$$\mu_2(I([T])) = 0 ,$$

in contradiction to

$$\mu_2(I([T])) = \mu_1([T]) > 0 .$$

Thus we have proved (3.6), which together with (3.4) and (3.5) implies (3.1).

Now let $\mathcal{H}'$ be a further finite family of half-spaces such that

$$[P] = \bigcap [\mathcal{H}']$$

holds. Then there exist formulas analogous to (3.5) and (3.6) with $\mathcal{H}'$ instead of $\mathcal{H}$, which together with (3.5) and (3.6) imply

$$\bigcap I([\mathcal{H}]) = \bigcup I([\mathcal{S}]) = \bigcap I([\mathcal{H}']) .$$

Thus we conclude that given $[P] \in (\mathcal{P}, \vartheta_{\mu_1})$, we can define the extension I' of $I \mid_{(\mathcal{Q}, \vartheta_{\mu_1})}$ by

$$I'([P]) := \bigcap I([\mathcal{H}]) ,$$

where $\mathcal{H}$ is an arbitrary finite family of half-spaces, for which $[P] = [\mathcal{H}]$ holds, independent of the special choice of $\mathcal{H}$.

So far we have shown that I' is a measure- and inclusion preserving mapping. We now show that I' is also injective.
Let P, P' be two polyhedra and w.l.o.g. $[P] \not\subseteq [P']$. Then there exists a half-space H, such that

$$[P'] \subseteq [H] \tag{3.9}$$

and $[P] \not\subseteq [H]$; i.e.

$$[P] \cap [\mathbb{E}^d \setminus H] \neq [\emptyset] \tag{3.10}$$

holds. Since I' preserves inclusions, we infer from (3.10)

$$I'([P]) \cap I'([\mathbb{E}^d \setminus H]) \neq [\emptyset]$$

and further

$$I'([P]) \not\subseteq I'([H]) . \tag{3.11}$$

From (3.9) we obtain (since I' is inclusion preserving)

$$I'([P']) \subseteq I'([H]) ,$$

which together with (3.11) implies

$$I'([P]) \neq I'([P]) .$$

Since $[P] \neq [P']$ were arbitrarily chosen, we conclude that I' is injective and thus we may define I'^{-1} on the image of I'.

Further we show that I'^{-1} also preserves inclusions.
We argue indirectly. If this were not true then there would exist polyhedra P, P' such that

(3.12) $$[P]_2 \subseteq [P']_2$$

as well as

(3.13) $$I'^{-1}([P]_2) \not\subseteq I'^{-1}([P']_2)$$

were fulfilled. (3.13) implies the existence of a half-space H which fulfills

(3.14) $$I'^{-1}([P']_2) \subseteq [H]$$

and

(3.15) $$I'^{-1}([P]_2) \cap [\mathbb{E}^d \setminus H] \neq [\emptyset] \ .$$

From (3.15) and the fact that I' is measure- and inclusion preserving, we get

$$I'(I'^{-1}([P]_2)) \cap I'([\mathbb{E}^d \setminus H]) \supseteq I'(I'^{-1}([P]_2) \cap [\mathbb{E}^d \setminus H]) \neq [\emptyset]_2$$

which implies together with

$$I'([H]) = [\mathbb{E}^d]_2 \setminus I'([\mathbb{E}^d \setminus H])$$

(that follows from (1.1)) finally

(3.16) $$[P]_2 \setminus I'([H]) \neq [\emptyset] \ .$$

Since I is inclusion preserving, therefore (3.16) and (3.14) together imply

$$[P]_2 \not\subseteq I'([H]) \supseteq I'(I'^{-1}([P']_2)) = [P']_2 \, ,$$

i.e.,

$$[P]_2 \not\subseteq [P']_2 \, ,$$

which contradicts (3.12). Thus we have shown that I'^{-1} preserves inclusions.

Thus I' is a measure- and inclusion preserving mapping whose inverse I'^{-1} is also inclusion preserving; i.e. I' is an isometry.

Finally we show that

$$I' : (\mathcal{P}, \vartheta_{\mu_1}) \to (\mathcal{P}, \vartheta_{\mu_2})$$

must be surjective and thus a bijection. So let P be an arbitrary polyhedron; then there exists a finite family of half-spaces $\mathcal{H}$ with $[P]_2 = \bigcap [\mathcal{H}]_2$. Further there exists a polyhedron Q, with $[Q]_1 := \bigcap I'^{-1}([\mathcal{H}]_2) = \bigcap I^{-1}([\mathcal{H}]_2)$ and thus we get

$$I'([Q]_1) = \bigcap I'(I'^{-1}([\mathcal{H}]_2)) = \bigcap [\mathcal{H}]_2 = [P]_2 \ ;$$

i.e., since P was arbitrarily chosen, I' is surjective and thus a bijection.

Since any extension $\tilde{I}$ of $I \mid_{(\mathcal{H},\vartheta_{\mu_1})}$ is measure preserving by the argument following (2.11), we obtain by the argument which proved (3.1*) that for all $[P] \in (\mathcal{P}, \vartheta_{\mu_1})$ on which an isometric extension $\tilde{I}$ is defined

$$[P] = \bigcap [\mathcal{H}] \ \Rightarrow \ \tilde{I}([P]) = \bigcap \tilde{I}([\mathcal{H}])$$

holds. So our construction of I' gives the only isometric extension of I to $\mathcal{P}$.

<u>4^{th} Part:</u> *We extend our isometry I' to the space of equivalence classes of all closed convex subsets of $\mathbb{E}^d$ which we denote by $(\mathcal{C}, \vartheta_{\mu_1})$.*

Let C be a closed convex set. We show that there exists a sequence $\langle P_n \rangle_{n\in\mathbf{N}}$ of polyhedra containing C such that

$$\lim_{n\to\infty} \mu_1(P_n) = \mu_1(C) \ .$$

We argue indirectly. If no such sequence exists then there exists an $\varepsilon > 0$ such that for any decreasing sequence $\langle R \rangle_{n\in\mathbf{N}}$ of polyhedra containing C

$$\lim_{n\to\infty} \mu_1(R_n) \geq \mu_1(C) + \varepsilon$$

holds and for some sequence $\langle P_n \rangle_{n\in\mathbf{N}}$ equality is attained. Thus $B := \bigcap_{n\in\mathbf{N}} P_n$ strictly contains C and further there exists $p \in tr(B, \mu_1)$, $p \notin C$. Since C is convex there exists a (closed) half-space H containing C such that ∂H strictly separates C and p. Thus we get for the sequence $\langle H \cap P_n \rangle_{n\in\mathbf{N}}$ of polyhedra

$$\lim_{n\to\infty} \mu_1(H \cap P_n) < \mu_1(C) + \varepsilon$$

which contradicts the assumption. Therefore there exists a sequence $\langle P_n \rangle_{n\in\mathbf{N}}$ with the asserted property.

Now we extend the isometry I' to the space $(\mathcal{C}, \vartheta_{\mu_1})$ and denote this extension like the original isometry by I; i.e. we define

$$I([C]) := \bigcap_{n\in\mathbf{N}} I'([P_n]) \ ,$$

where $\langle P_n \rangle_{n\in\mathbf{N}}$ denotes an arbitrary sequence of polyhedra containing C with

$$\lim_{n\to\infty} \mu_1(P_n) = \mu_1(C) \ .$$

We show that I is well-defined. Therefore we assume w.l.o.g. that the sequence $\langle P_n\rangle_{n\in\mathbf{N}}$ is monotonically decreasing and choose for any such sequence a convex set $C_{\langle P_n\rangle}$ for which

$$[C_{\langle P_n\rangle}]_2 = \bigcap_{n\in\mathbf{N}} I'([P_n])$$

holds. Since then $\langle I'([P_n])\rangle_{n\in\mathbf{N}}$ is also monotonically decreasing and $\mu_1(P_n) = \mu_2(I'([P_n]))$ we get

$$\mu_2([C_{\langle P_n\rangle}]_2) = \mu_1(C) \; ;$$

i.e., the measure is independent of the choice of the sequence $\langle P_n\rangle_{n\in\mathbf{N}}$.

(4.1) Thus we know that I is measure- and inclusion preserving if it is well-defined.

Let $\langle P_n\rangle_{n\in\mathbf{N}}$ and $\langle Q_n\rangle_{n\in\mathbf{N}}$ be sequences which decrease towards C; then $\langle P_n \cap Q_n\rangle_{n\in\mathbf{N}}$ also decreases towards C. Since I' preserves inclusions we get

$$[C_{\langle P_n\rangle}]_2 \supseteq [C_{\langle P_n\cap Q_n\rangle}]_2$$

and

$$[C_{\langle Q_n\rangle}]_2 \supseteq [C_{\langle P_n\cap Q_n\rangle}]_2 \; .$$

Since

$$\mu_2([C_{\langle P_n\rangle}]_2) = \mu_2([C_{\langle P_n\cap Q_n\rangle}]_2) = \mu_2([C_{\langle Q_n\rangle}]_2)$$

we obtain

$$[C_{\langle P_n\rangle}]_2 = [C_{\langle P_n\cap Q_n\rangle}]_2 = [C_{\langle Q_n\rangle}]_2 \; .$$

So I is well-defined on $(\mathcal{C}, \vartheta_{\mu_1})$ and thus by (4.1) measure- and inclusion preserving.

Analogously to the proof of the 3^{rd} part we see that I is an isometry and our construction gives the only isometric extension of I' onto $(\mathcal{C}, \vartheta_{\mu_1})$. Thus it is clear that the extension I coincides with the original mapping on $(\mathcal{D}, \vartheta_{\mu_2})$.

Finally it remains to prove that I is a surjection onto $(\mathcal{C}, \vartheta_{\mu_2})$. We show that if C is a closed convex set then $I^{-1}([C]_2)$ exists. Let $\langle P_n\rangle_{n\in\mathbf{N}}$ be a sequence of polyhedra which decreases monotonically to C. Since I'^{-1} preserves inclusions, $\langle I'^{-1}([P_n]_2)\rangle_{n\in\mathbf{N}}$ is also monotonically decreasing towards an equivalence class $[C']$ of a closed convex set C' for which

$$I([C']) = I(\bigcap_{n\in\mathbf{N}} I'^{-1}([P_n]_2)) = \bigcap_{n\in\mathbf{N}} I'(I'^{-1}([P_n]_2)) = \bigcap_{n\in\mathbf{N}} [P_n]_2 = [C]_2$$

holds; i.e. $[C]_2$ possesses indeed the pre-image $I^{-1}([C]_2) = [C']$.

5th Part: *Construction of the sets A_i and the mapping ϕ;*
ϕ is a measure preserving homeomorphism inducing I.

Thus let for any $n \in \mathbf{N}$

(5.1) $$\begin{cases} K_n \text{ be the smallest compact set with} \\ \qquad K_n \in [nB^d] \cap I^{-1}([nB^d]_2) \end{cases}$$

and

(5.2) $$\begin{cases} L_n \text{ the smallest compact set with} \\ \qquad L_n \in I([nB^d]) \cap [nB^d]_2 \,. \end{cases}$$

(Since the topology of $\mathbb{E}^d$ possesses a countable basis there exists in each equivalence class $[M]_i$ of a closed set M a smallest closed set M^* (with respect to set-theretic inclusion). The compactness of K_n and L_n follows from $K_n, L_n \subseteq nB^d$.) We have now

(5.3) $$K_n \subseteq K_{n+1}, L_n \subseteq L_{n+1}$$

and

(5.4) $$[conv(K_n)] = [K_n] \text{ as well as } [conv(L_n)] = [L_n]\,.$$

Since I and I^{-1} preserve inclusions we have

(5.5) $$I([K_n]) = [L_n]_2$$

and since $\lim_{n\to\infty} \mu_1(K_n) = \lim_{n\to\infty} \mu_2(L_n) = 1$ we obtain

$$\mu_1(A_1) = \mu_2(A_2) = 1$$

if we set

(5.6) $$A_1 := \bigcup_{n\in\mathbf{N}} K_n \text{ and } A_2 := \bigcup_{n\in\mathbf{N}} L_n\,.$$

Further we have

(5.7) $$K_n = tr(K_n, \mu_1) = K_n \cap tr(\mathbb{E}^d, \mu_1)$$

as well as

(5.8) $$L_n = tr(L_n, \mu_2) = L_n \cap tr(\mathbb{E}^d, \mu_2)$$

and thus

$$A_i \subseteq tr(\mathbb{E}^d, \mu_i)\,.$$

Now we show that

$$int(conv(tr(\mathbb{E}^d,\mu_i)))\cap tr(\mathbb{E}^d,\mu_i)\subseteq A_i\ ,$$

i.e., if

$$p\in int(conv(tr(\mathbb{E}^d,\mu_1)))\cap tr(\mathbb{E}^d,\mu_1)\,,$$

then there exists an $n\in\mathbf{N}$ with $p\in K_n$.

We argue indirectly: if p is in none of the compact sets K_n then we show that p is also not in $int(conv(K_n))$ (which is a subset of $int(conv(tr(\mathbb{E}^d,\mu_1)))$).

In fact, if $p\in int(conv(K_n))$ held then this would imply that there exists a neighbourhood $U_n(p)$ such that $U_n(p)\subset conv(K_n)$. Since $p\notin K_n$, we also find a neighbourhood $V_n(p)$ such that $V_n(p)\cap K_n=\emptyset$ and thus

(5.9) $$V_n(p)\cap U_n(p)\subset conv(K_n)\ \text{ and }\ V_n(p)\cap U_n(p)\cap K_n=\emptyset\ .$$

Since $p\in int(tr(\mathbb{E}^d,\mu_1))$ we deduce $\mu_1(V_n(p)\cap U_n(p))>0$ and together with (5.9)

$$\mu_1(conv(K_n))>\mu_1(K_n)\,,$$

which contradicts (5.4). So we have shown that $p\notin int(conv(K_n))$.

Therefore we have

$$p\notin int(\bigcup_{n\in\mathbf{N}}conv(K_n))\ .$$

Since $int(\bigcup_{n\in\mathbf{N}}conv(K_n))$ is by (5.3) and by the monotonicity of conv(.) convex, there exists a hyperplane F containing p which does not intersect $int(\bigcup_{n\in N}conv(K_n))$. Since $p\in int(conv(tr(\mathbb{E}^d,\mu_1)))$ we see further that there exists $q\in tr(\mathbb{E}^d,\mu_1)$ such that

$$q\notin cl(\bigcup_{n\in\mathbf{N}}K_n)\ .$$

Thus there exists a neighbourhood $U(q)$ of q with $\mu_1(U(q))>0$ which does not intersect the set $A_1=\bigcup_{n\in\mathbf{N}}K_n$ in contradiction to $\mu_1(A)=1$.

So hereby and by the analogous reasoning for

$$p\in int(conv(tr(\mathbb{E}^d,\mu_2)))\cap tr(\mathbb{E}^d,\mu_2)$$

we obtain

$$int(conv(tr(\mathbb{E}^d,\mu_i)))\cap tr(\mathbb{E}^d,\mu_i)\subseteq A_i\ .$$

If the sets $tr(\mathbb{E}^d,\mu_1)$ and $tr(\mathbb{E}^d,\mu_2)$ are bounded and thus compact we get even

$$A_i=tr(\mathbb{E}^d,\mu_i)\,,$$

since by (5.1) and (5.2)

$$K_n = tr(\mathbb{E}^d, \mu_1) \text{ and } L_n = tr(\mathbb{E}^d, \mu_2)$$

for suitable n.

For simplicity of notation we introduce functions

$$J : \mathcal{C}_1 \to \mathcal{C}_2 \text{ and } J_{-1} : \mathcal{C}_2 \to \mathcal{C}_1 \ ,$$

where

$$C \in \mathcal{C}_i \ :\Leftrightarrow \ [C]_i \in (\mathcal{C}, \vartheta_{\mu_i})$$

by:
$J(C)$ [resp. $J_{-1}(C)$] is the smallest closed set contained in $I([C])$ [resp. $I^{-1}([C])$]. (Such sets exist, since the topology of $\mathbb{E}^d$ possesses a countable basis.) Further we get:

(5.10) $[J(C)]_2 = I([C])$ as well as $[J_{-1}(C)] = I^{-1}([C]_2)$ for all $C \in \mathcal{C}_1$ respectively $C \in \mathcal{C}_2$.

Further since I and I^{-1} are by Proposition 1 inclusion preserving, we have:

(5.11) The functions J and J_{-1} are inclusion preserving set-mappings.

The fact that $J(C)$ and $J_{-1}(C)$ are minimal closed sets in their equivalence classes can also be described in the form

(5.12) $$J(C) = tr(J(C), \mu_2) \text{ and } J_{-1}(C) = tr(J_{-1}(C), \mu_1) \ .$$

Now let $x \in A_1$. There exists by (5.6) an $n \in \mathbf{N}$ with $x \in K_n$. Further let

(5.13) $\langle C_m \rangle_{m \in \mathbf{N}}$ be a sequence of compact sets which decreases monotonically to x such that $\mu_1(C_m) > 0$, $C_m \subseteq K_n$ and $x \in tr(C_m, \mu_1)$. (Such a sequence exists by (5.7).)

(5.14) We show next that $\bigcap_{m \in \mathbf{N}} J(C_m)$ contains one and only one point.

$C := \bigcap_{m \in \mathbf{N}} J(C_m)$ is non-empty, since $\langle J(C_m) \rangle_{m \in \mathbf{N}}$ is by (5.11) and (5.13) a monotonically decreasing sequence of compact non-empty sets. To show that C contains only one point we proceed indirectly:

Let p, q be two different points in C and thus in $tr(J(C_m), \mu_2) = J(C_m)$. We choose an $\varepsilon > 0$ so that if we define compact sets $D_p, D_q \subseteq J(C_1)$ by

(5.15) $$D_p := \varepsilon B(p) \cap J(C_1) \text{ and } D_q = \varepsilon B(q) \cap J(C_1)$$

we get

$$conv(D_p) \cap conv(D_q) = \emptyset \ .$$

(5.16) First we show that the sets $J_{-1}(D_p)$ and $J_{-1}(D_q)$ contain x.

We argue indirectly: in the contrary case there exists an m such that C_m does not meet one of these sets. W.l.o.g we may assume $J_{-1}(D_p) \cap C_m = \emptyset$. Thus, since I is a measure preserving isometry we get

$$I([J_{-1}(D_p)]) \cap I([C_m]) = [\emptyset]$$

and by (5.10)

$$\mu_2(D_p \cap J(C_m)) = 0 \ .$$

But this contradicts

$$p \in tr(J(C_m), \mu_2) \ ,$$

since by (5.15) $D_p \cap J(C_m)$ is a neighbourhood of p in $J(C_m) = tr(J(C_m), \mu_2)$. Thus we have shown $x \in J_{-1}(D_p), J_{-1}(D_q)$; i.e., we have proved (5.16).

By (5.16) and (5.12) the (possibly) empty set $\mathcal{G}_1$ of equivalence classes containing a closed half-space H with

$$[J_{-1}(D_p)] \subseteq [H] \ \text{ and } \ [J_{-1}(D_q)] \cap [H] = [\emptyset]$$

is contained in the set of equivalence classes containing a closed half-space with boundary hyperplane through x. Each closed half-space H_z with boundary hyperplane through x is completely determined by its interior normal vector $z \in S^{d-1}$. Further, since μ_1 varies by (2) continuously on half-spaces, the function $\zeta(z) := \mu_1(H_z)$ is continuous in z with range $[0, 1]$ and thus the mapping

$$\iota : S^{d-1} \mapsto S^{d-1} \times [0, 1] \qquad \iota(z) = (z, \zeta(z))$$

is a homeomorphism onto its image. If we let $M \subseteq S^{d-1}$ be the space of all z with $\iota(z) \in (0, 1)$ then $dim(M) \leq d - 1$. Further the mapping f_1 constructed in the 2^{nd} part of the proof of this theorem is a homeomorphism from $S^{d-1} \times (0, 1)$ onto its image, which is a subset of $(\mathcal{H}, \vartheta_{\mu_1})$. Thus also

$$f_1 \circ \iota : M \mapsto (\mathcal{H}, \vartheta_{\mu_1}) \quad z \mapsto [H_z]$$

is a homeomorphism onto its image and the dimension of this image $dim(Im(f_1 \circ \iota(M)))$ is smaller or equal to $d - 1$. Since $\mathcal{G}_1 \subseteq f_1(\iota(M))$ ($\mathcal{G}_1$ contains only equivalence classes $[H_z]$ such that $0 < \mu_1([H_z]) < 1$) we get

$$dim(\mathcal{G}_1, \vartheta_{\mu_1}) \leq d - 1 \ .$$

Let $\mathcal{G}'$ be the set of all half-spaces such that

$$conv(D_p) \subseteq H \text{ and } D_q \cap H = \emptyset \, .$$

Then $dim(\mathcal{G}') = d$ since $conv(D_p)$ and $conv(D_q)$ are compact convex subsets with empty intersection. The set $\mathcal{G}_2$ of equivalence classes containing a closed half-space H such that

$$[D_p]_2 \subseteq [H]_2 \text{ and } [D_q]_2 \cap [H]_2 = [\emptyset]$$

contains the set of all equivalence classes $[\mathcal{G}']$ of closed half-spaces $H \in \mathcal{G}'$. Since $\mu_2(conv(D_p)), \mu_2(conv(D_q)) > 0$ the mapping

$$[.]_2 : \mathcal{G}' \mapsto [\mathcal{G}']_2 \subset (\mathcal{H}, \vartheta_{\mu_2})$$

is by (3) a bijection and by (2) this mapping is also continuous. Thus $dim([\mathcal{G}'], \vartheta_{\mu_2}) \geq dim(\mathcal{G}') = d$ and since $[\mathcal{G}'] \subseteq \mathcal{G}_2$ we get

$$dim(\mathcal{G}_2, \vartheta_{\mu_2}) \geq d \, .$$

By (5.10) and since I^{-1} preserves inclusions, the isometry I^{-1} maps the space $(\mathcal{G}_2, \vartheta_{\mu_2})$ with dimension $\geq d$ homeomorphically into the space $(\mathcal{G}_1, \vartheta_{\mu_1})$ with dimension $\leq d-1$ which is a contradiction. Thus $\bigcap_{m \in \mathbf{N}} C_m$ contains one and only one point; i.e. (5.14) is proved.

Now we define a map $\phi : A_1 \to A_2$ (that will be proved to be a homeomorphism) by

$$\{\phi(x)\} := \bigcap_{m \in \mathbf{N}} J(C_m)$$

if $\langle C_m \rangle_{m \in \mathbf{N}}$ fulfills the conditions of (5.13).

$\phi(x)$ is well-defined, since if the sequences $\langle C_m \rangle_{m \in \mathbf{N}}$ and $\langle C'_m \rangle_{m \in \mathbf{N}}$ fulfil the conditions of (5.13), the sequence

$$\langle conv(C_m \cup C'_m) \cap K_n \rangle_{m \in \mathbf{N}}$$

also fulfills them. By (5.11) we get

$$\bigcap_{m \in \mathbf{N}} J(C_m) \subseteq \bigcap_{m \in \mathbf{N}} J(conv(C_m \cap C'_m) \cap K_n) \supseteq \bigcap_{m \in \mathbf{N}} J(C'_m)$$

and since each of the intersections contains one and only one point

$$\bigcap_{m \in \mathbf{N}} J(C_m) = \bigcap_{m \in \mathbf{N}} J(C'_m) \, ,$$

i.e., $\bigcap_{m\in\mathbf{N}} J(C_m)$ is independent of $\langle C_m\rangle_{m\in\mathbf{N}}$ and thus $\phi(x)$ is well-defined.

So it remains only to prove that ϕ^{-1} exists and that ϕ as well as ϕ^{-1} is continuous.

First we show the existence of ϕ^{-1}. Let $y \in A_2$. We define in analogy with (5.13) a sequence $\langle D_n\rangle_{m\in\mathbf{N}}$ of compact sets contained in L_n which converges to y such that $y \in tr(D_m, \mu_2)$. Analogously to the previous considerations we get that

$$\{\psi(x)\} := \bigcap_{m\in\mathbf{N}} J_{-1}(D_m)$$

defines a mapping from A_2 into A_1.

Since J_{-1} is the inverse of J on the set of minimal closed representatives of the equivalence classes with respect to ϑ_{μ_i}, we get

$$\psi = \phi^{-1}$$

in the following way. Let $\langle C_m\rangle_{m\in\mathbf{N}}$ be a sequence, which fulfills (5.13) and for which further $C_m = tr(C_m, \mu_2)$ holds. Then

$$\{\phi(x)\} = \bigcap_{m\in\mathbf{N}} J(C_m)$$

and the sequence $\langle J(C_m)\rangle_{m\in\mathbf{N}}$ satisfies conditions analogous to (5.13). So, if we set w.l.o.g. $D_m := J(C_m)$ in the definition of ψ, we get

$$\{\psi(\phi(x))\} = \bigcap_{m\in\mathbf{N}} J_{-1}(J(C_m))$$

and since J_{-1} is inverse to J on the set of minimal closed representatives of the equivalence classes with respect to ϑ_{μ_i}, we conclude that

$$\{\psi(\phi(x))\} = \bigcap_{m\in\mathbf{N}} C_m = \{x\}\ ;$$

i.e., $\psi(\phi(x)) = x$ for all $x \in A_1$. Analogously we obtain $\phi(\psi(y)) = y$ for all $y \in A_2$ and thus $\psi = \phi^{-1}$.

Further we show

(5.17) $$M \subseteq K_j \text{ and } M = tr(M, \mu_1) \Rightarrow \phi(M) = J(M)$$

and

(5.18) $$M \subseteq L_j \text{ and } M = tr(M, \mu_2) \Rightarrow \phi^{-1}(M) = J_{-1}(M)\ .$$

We argue indirectly: By definition of ϕ we have $\phi(M) \subseteq J(M)$. Assume that there exists
$$p \in J(M) \setminus \phi(M) \ ,$$
then we get by the definition of the inverse $\psi = \phi^{-1}$
$$q := \phi^{-1}(p) \in J_{-1}(J(M)) = M \ .$$
and thus $\phi(q) = p$ in contradiction to our assumption. So (5.17) has been proved; analogously (5.18) may be shown.

Now we show the continuity of ϕ:

(5.19) Let $x \in A_1$ and $\langle x_m \rangle_{m \in \mathbf{N}}$ an arbitrary sequence of points in A_1 converging to x. We distinguish two cases:

1^{st} There exists an $n \in \mathbf{N}$ with $\langle x_m \rangle_{m \in \mathbf{N}} \subset K_n$.

2^{nd} There exists no such n.

(5.20) In the first case we let
$$C_l := (\bigcup_{m \geq l} (x_m + \frac{1}{m} B^d) \cup x) \cap K_n \ .$$
Then $\langle C_l \rangle_{l \in \mathbf{N}}$ is a sequence of compact sets monotonically decreasing to x, which fulfills the conditions of (5.13). Thus by (5.11) and (5.14) also the sequence $\langle J(C_l) \rangle_{l \in \mathbf{N}}$ decreases monotonically to $\phi(x)$. Since $\phi(x_l) \in J(C_l)$, and $J(C_l)$ is compact we see that $\langle \phi(x_l) \rangle_{l \in \mathbf{N}}$ converges to $\phi(x)$; in this case the continuity of ϕ is proved.

(5.21) Next we show that the second case of (5.19) does not occur.

We argue indirectly: First we show that if we assume case 2 in (5.19) then

(5.22) the sequence $\langle \phi(x_m) \rangle_{m \in \mathbf{N}}$ is unbounded.

In fact else the sequences $\langle x_m \rangle_{m \in \mathbf{N}}$ and $\langle \phi(x_m) \rangle_{m \in \mathbf{N}}$ are bounded and thus there exists an $n \in \mathbf{N}$ such that $int(nB^d)$ contains both sequences. Now let x_m be an arbitrary given point, then there exists by (5.6) a $j \in \mathbf{N}$ with $\phi(x_m) \in L_j$. Thus
$$\phi(x_m) \in int(nB^d) \cap L_j \ .$$
Since $int(nB^d)$ is open and by (5.1) L_j is minimal closed in $[L_j]_2$, i.e. $L_j =$

$tr(L_j, \mu_2)$ (cf. (5.8)), we infer from this

$$\phi(x_m) \in tr((int(nB^d) \cap L_j), \mu_2)$$

and thus

$$x_m = \phi^{-1}(\phi(x_m)) \in J^{-1}(tr((int(nB^d) \cap L_j), \mu_2)) \ .$$

Since we have assumed $x_m \in int(nB^d)$, we get

$$x_m \in int(nB^d) \cap J^{-1}(tr((int(nB^d) \cap L_j), \mu_2)) \ ,$$

which implies further

$$x_m \in tr((int(nB^d) \cap J^{-1}(tr((int(nB^d) \cap L_j), \mu_2))), \mu_1) \ , \tag{5.23}$$

since $int(nB^d)$ is open and $J^{-1}(tr((int(nB^d) \cap L_j), \mu_2))$ is minimal closed in its equivalence class (i.e., since by (5.12)

$$J^{-1}(tr((int(nB^d) \cap L_j), \mu_2)) = tr(J^{-1}(tr((int(nB^d) \cap L_j), \mu_2)), \mu_1)$$

holds). We have

$$[int(nB^d)] \subseteq [nB^d]$$

and by (5.10)

$$[J^{-1}(tr((int(nB^d) \cap L_j), \mu_2)] \subseteq [J^{-1}(nB^d)] \subseteq I^{-1}([nB^d]_2) \ .$$

From these, we obtain

$$[tr((int(nB^d) \cap J^{-1}(tr((int(nB^d \cap L_j), \mu_2))), \mu_1)] \subseteq [nB^d] \cap I^{-1}([nB^d]_2)$$

and thus by (5.1)

$$tr((int(nB^d) \cap J^{-1}(tr((int(nB^d) \cap L_j), \mu_2))), \mu_1) \subseteq K_n \ ,$$

and further by (5.23)

$$x_m \in K_n \ ,$$

in contradiction to the 2^{nd} case of (5.19). Thus we have proved (5.22).

Therefore the sequence $\langle \phi(x_m) \rangle_{m \in \mathbf{N}}$ is unbounded and so, since S^{d-1} is compact, we may choose a subsequence (which we also denote by $\langle x_m \rangle_{m \in \mathbf{N}}$) such that

$$\phi(x_m) \notin mB^d$$

and such that there exists a $v \in S^{d-1}$ with

$$\lim_{m \to \infty} \frac{\phi(x_m)}{\|\phi(x_m)\|} = v \ .$$

Now we choose $n \in \mathbf{N}$ so that $x \in K_n$. Then there exists an index k and a closed convex set K, such that for all $m \geq k$

$$\phi(x_m) \in int(K) \ \text{ and } \ nB^d \cap K = \emptyset \tag{5.24}$$

holds. Since there exists for any m a j_m with $\phi(x_m) \in L_{j_m}$ and since $\phi(x_m) \in int(K)$, there exists a set $M_m \subseteq L_{j_m} \cap int(K)$ such that $\phi(x_m) \in M_m$ and $M_m = tr(M_m, \mu_2)$. Application of (5.18) gives now

$$x_m = \phi^{-1}(\phi(x_m)) \in \phi^{-1}(M_m) = J_{-1}(M_m) =$$

$$= tr(J_{-1}(M_m), \mu_1) = tr(\phi^{-1}(M_m), \mu_1) \subseteq tr(\phi^{-1}(K), \mu_1) \ ,$$

i.e.,

$$x_m \in tr(\phi^{-1}(K), \mu_1)) \,.$$

Since $\langle x_m \rangle_{m \in \mathbf{N}}$ converges to x we get further

$$x \in tr(\phi^{-1}(K), \mu_1)) \,. \tag{5.25}$$

Next we apply a dimension argument analogous to (5.14).

We consider the (possibly empty) set $\mathcal{G}_1$ of equivalence classes containing a half-space H such that

$$[K_n] \subseteq [H] \ \text{ and } [\phi^{-1}(K)] \cap [H] = [\emptyset] \ .$$

Since by (5.25)

$$x \in K_n \cap tr(\phi^{-1}(K), \mu_1)$$

holds, we get by (5.7) that $\mathcal{G}_1$ is contained in the set of all equivalence classes containing a half-space whose boundary hyperplane contains x. We obtain, following the lines of the proof of (5.14), that $dim(\mathcal{G}_1) \leq d - 1$.

We let $\mathcal{G}_2$ be the set of all equivalence classes containing a half-space H such that

$$[\phi(K_n)]_2 \subseteq [H]_2 \ \text{ and } \ [K]_2 \cap [H]_2 = [\emptyset] \ .$$

Since by (5.2) $\phi(K_n) = L_n \subseteq nB^d$ holds, we see that $\mathcal{G}_2$ contains the set of all equivalence classes $[\mathcal{G}']$ of closed half-spaces $H \subseteq \mathcal{G}'$ such that $nB^d \subset H$ and $K \cap H = \emptyset$. Since K is a closed convex set and by (5.24) $nB^d \cap K = \emptyset$ we conclude that $\mathcal{G}'$ is d-dimensional and thus $dim(\mathcal{G}_2) \geq d$ (again following the lines of the proof of (5.14)).

Since I^{-1} is an inclusion preserving isometry it maps $\mathcal{G}_2$ homeomorphically into $\mathcal{G}_1$. Thus the dimension of $\mathcal{G}_1$ cannot be smaller than that of $\mathcal{G}_2$. However this contradicts $dim\mathcal{G}_1 < d \leq dim\mathcal{G}_2$, and so (5.21) is proved.

Thus only the 1^{st} case in (5.19) can occur. This implies together with (5.20) the continuity of ϕ.

The continuity of ϕ^{-1} may be shown completely analogously.

Now we are able to show that the homeomorphism ϕ induces the isometry I and that ϕ as well as ϕ^{-1} are measure preserving mappings.

First we show that

$$\phi(L \cap A_1) = N \cap A_2 \tag{5.26}$$

if $J(L) = N$ and $J_{-1}(N) = L$; i.e., if L and N are the minimal closed representatives of their equivalence classes.

Let $x \in L \cap A_1$. Since we know by the proof of the continuity of ϕ that only case 1 in (5.19) can occur, there exists a neighbourhood $U(x)$ and an $n \in \mathbf{N}$ such that

$$U(x) \cap A_1 \cap L \subset K_n \cap L \ .$$

Since L is minimal closed in its equivalence class we see that

$$x \in tr(K_n \cap L, \mu_1) \ .$$

By definition of ϕ and J we obtain

$$\phi(x) \in J(tr(K_n \cap L, \mu_1)) \cap A_2 \subseteq J(L) \cap A_2 = N \cap A_2 \ ,$$

and further, since $x \in L \cap A_1$ was arbitrarily chosen,

$$\phi(L \cap A_1) \subseteq N \cap A_2 \ .$$

Together with the analogous formula

$$\phi^{-1}(N \cap A_2) \subseteq L \cap A_1$$

we get the validity of (5.26).

Next we show that for sets M with $M \subseteq A_1$ and $[M] \in (\mathcal{C}, \vartheta_{\mu_1})$ in fact

$$[\phi(M)]_2 = I([M])$$

holds; i.e., I is induced by ϕ.

By (5.26) and (5.10) we get

$$[\phi(L \cap A_1)]_2 = [N \cap A_2]_2 = [J(L) \cap A_2]_2 = I([L]) \tag{5.27}$$

for L with $L = tr(L, \mu_1)$ and $[L] \in (\mathcal{C}, \vartheta_{\mu_1})$. Let W be an open cube. Then we see

$$W \cap A_1 \subseteq tr(W, \mu_1) \cap A_1 \quad \text{and} \quad [tr(W, \mu_1)] = [W] \ ,$$

since by (2) μ_1 vanishes on hyperplanes. If we set now in (5.27) $L = tr(W, \mu_1)$ we obtain

$$[\phi(W \cap A_1)]_2 \subseteq [\phi(tr(W, \mu_1) \cap A_1)]_2 = I([tr(W, \mu_1)] = I([W]) \ . \tag{5.28}$$

Now let M be an arbitrary set with $M \subseteq A_1$ and $[M] \in (\mathcal{C}, \mu_1)$, and let $L = tr(M, \mu_1)$. Then we get

$$\mu_1(M \triangle L) = 0 \ .$$

So, since μ_1 is a Baire measure and thus regular and since the family of open cubes constitutes a basis of the topology of $\mathbb{E}^d$, we can find for a given $\varepsilon > 0$ a countable family of open cubes $\mathcal{W}_\varepsilon$ such that

$$M \triangle L \subseteq \bigcup \mathcal{W}_\varepsilon \quad \text{and} \quad \sum_{W \in \mathcal{W}_\varepsilon} \mu_1(W) < \varepsilon \tag{5.29}$$

holds. From (5.28) and (5.29) we conclude

$$[\phi((M \triangle L) \cap A_1)]_2 \subseteq [\phi(\bigcup \mathcal{W}_\varepsilon \cap A_1)]_2 =$$

$$= \bigcup_{W \in \mathcal{W}_\varepsilon} [\phi(W \cap A_1)]_2 \subseteq \bigcup_{W \in \mathcal{W}_\varepsilon} I([W]) \ ,$$

and further, since I is measure preserving, we obtain by (5.29)

$$\mu_2(\phi(L \triangle M)) \leq \mu_2(\bigcup_{W \in \mathcal{W}_\varepsilon} I([W])) \leq$$

$$\leq \sum_{W \in \mathcal{W}_\varepsilon} \mu_2(I([W])) = \sum_{W \in \mathcal{W}_\varepsilon} \mu_1(W) < \varepsilon \ .$$

Since this holds for any $\varepsilon > 0$, we see that

$$\mu_2(\phi(L \triangle M)) = 0$$

and thus, since $L = tr(L, \mu_1)$, we get by (5.27) and (5.29)

$$[\phi(M)]_2 = [\phi(L \cap A_1)]_2 = I([L]) = I([M]) \ .$$

We conclude that for arbitrary sets M with $M \subseteq A_1$ and $[M] \in (\mathcal{C}, \vartheta_{\mu_1})$

$$[\phi(M)]_2 = I([M])$$

holds; i.e., I is induced by ϕ, as asserted above.

Now we show that ϕ and ϕ^{-1} are measure preserving mappings.

The mappings ϕ and ϕ^{-1} are continuous, hence Baire measurable. Since ϕ induces the measure preserving mapping I, and $\mu_1(C \cap A_1) = \mu_2(\phi(C \cap A_1))$ as well as $\mu_2(C \cap A_2) = \mu_1(\phi^{-1}(C \cap A_2))$ for any closed convex set C - and thus in particular for any closed cube W- and μ_1 and μ_2 vanish on hyperplanes, therefore we get for any open cube W

$$\mu_2(\phi(W \cap A_1)) = \mu_1(W) \text{ and } \mu_1(\phi^{-1}(W \cap A_2)) = \mu_2(W) \,. \tag{5.30}$$

Since μ_1 and μ_2 are regular measures and the family of open cubes constitutes a basis of the topology of $\mathbb{E}^d$, we can find for any Baire measurable set $M \subseteq A_1$ and any $\varepsilon > 0$ a countable family of open cubes $\mathcal{W}^1_\varepsilon$ such that

$$M \subseteq \bigcup \mathcal{W}^1_\varepsilon \text{ and } \sum_{W \in \mathcal{W}^1_\varepsilon} \mu_1(W) \leq \mu_1(M) + \varepsilon \,,$$

as well as a countable family $\mathcal{W}^2_\varepsilon$ of open cubes such that

$$\phi(M) \subseteq \bigcup \mathcal{W}^2_\varepsilon \text{ and } \sum_{W \in \mathcal{W}^2_\varepsilon} \mu_2(W) \leq \mu_2(\phi(M)) + \varepsilon \,.$$

By (5.30) we see that for any $\varepsilon > 0$

$$\mu_2(\phi(M)) \leq \mu_2(\phi(\bigcup \mathcal{W}^1_\varepsilon \cap A_1)) \leq$$

$$\leq \sum_{W \in \mathcal{W}^1_\varepsilon} \mu_2(\phi(W \cap A_1)) = \sum_{W \in \mathcal{W}^1_\varepsilon} \mu_1(W) \leq \mu_1(M) + \varepsilon$$

and

$$\mu_1(M) = \mu_1(\phi^{-1}(\phi(M)) \leq \mu_1(\phi^{-1}(\bigcup \mathcal{W}^2_\varepsilon \cap A_1)) \leq$$

$$\leq \sum_{W \in \mathcal{W}^2_\varepsilon} \mu_1(\phi^{-1}(W \cap A_1)) = \sum_{W \in \mathcal{W}^2_\varepsilon} \mu_2(W) \leq \mu_2(\phi(M)) + \varepsilon \,.$$

Thus we obtain

$$\mu_2(\phi(M)) = \mu_1(M)$$

for any Baire measurable set $M \subseteq A_1$ and, since $\phi : A_1 \to A_2$ is a homeomorphism, we get also $\mu_1(\phi^{-1}(M)) = \mu_2(M)$ for any Baire measurable set $M \subseteq A_2$; i.e., ϕ and ϕ^{-1} are measure preserving.

6^{th} *Part:* *The dimension of the affine hull of a set remains invariant under ϕ.*

To show that ϕ does not change the dimension of the affine hull of a set, we prove first:

(6.1) ϕ maps subsets of hyperplanes contained in A_1 and ϕ^{-1} maps subsets of hyperplanes contained in A_2 into subsets of hyperplanes.

For this purpose we introduce the mapping J^* which maps any closed half-space H with $0 < \mu_1(H) < 1$ to the closed half-space $J^*(H) \in I([H])$ which is uniquely determined by (3) and the first part of the proof (since $0 < \mu_2(I([H])) < 1$).

Then we have:

(6.2) $$J(H) \subseteq J^*(H) ,$$

(6.3) J^* is a bijection which maps the set of closed half-spaces with μ_1-measure greater than 0 and smaller than 1 onto the set of all closed half-spaces with μ_2-measure greater than 0 and smaller than 1. (This follows by (3) and the 1^{st} part of the proof, since I is a measure preserving mapping.)

(6.4) J^* and J^{*-1} are measure- and inclusion preserving isometries.

Now we prove that

(6.5) $$0 < \mu_1(H) < 1,\ p \in H \cap A_1 \Rightarrow \phi(p) \in J^*(H)$$

as well as

(6.6) $$0 < \mu_2(H) < 1,\ p \in H \cap A_2 \Rightarrow \phi^{-1}(p) \in J^{*-1}(H)$$

holds. First we show

(6.7) $$0 < \mu_1(H) < 1,\ p \in int(H) \cap A_1 \Rightarrow \phi(p) \in J^*(H) .$$

If $p \in int(H)$ we get $p \in tr(H, \mu_1)$ and by (5.26), (5.11) and (6.2)

$$\phi(p) \in J(tr(H, \mu_1)) \cap A_2 \subseteq J(H) \subseteq J^*(H) ,$$

thus (6.7) is proved.

Next we show

$$0 < \mu_1(H) < 1,\ p \in \partial H \Rightarrow \phi(p) \in J^*(H)\ . \tag{6.8}$$

If $p \in \partial H$ then there is a sequence $\langle H_n \rangle_{n\in\mathbf{N}}$ of half-spaces converging to H such that $p \in int(H_n)$ and $0 < \mu_1(H_n) < 1$. This implies by (6.7)

$$\phi(p) \in J^*(H_n)\ ,$$

and by (6.4) we have the relations

$$\bigcap_{n\in\mathbf{N}} J^*(H_n) \supseteq J^*(H)$$

and

$$\mu_2(\bigcap_{n\in\mathbf{N}} J^*(H_n)) = \lim_{n\to\infty} \mu_2(J^*(H_n)) = \lim_{n\to\infty} \mu_1(H_n) =$$

$$= \mu_1(\bigcap_{n\in\mathbf{N}} H_n) = \mu_1(H) = \mu_2(J^*(H))\ .$$

From these we conclude by (3)

$$\bigcap_{n\in\mathbf{N}} J^*(H_n) = J^*(H)$$

and thus

$$p \in \partial H \ \Rightarrow\ \phi(p) \in \bigcap_{n\in\mathbf{N}} J^*(H_n) = J^*(H)\ ,$$

i.e. (6.8) holds. Now (6.8) and (6.7) imply (6.5). (6.6) can be proved analogously.

Next we show:

$$\text{(6.9)}\qquad \begin{cases} \text{If } h \text{ is a hyperplane such that} \\ \qquad h \cap A_1 \cap int(conv(tr(\mathbb{E}^d, \mu_1))) \neq \emptyset \\ \text{then the set } \phi(h \cap A_1) \text{ is contained in a hyperplane } g. \end{cases}$$

and

$$\text{(6.10)}\qquad \begin{cases} \text{If } h \text{ is a hyperplane such that} \\ \qquad h \cap A_2 \cap int(conv(tr(\mathbb{E}^d, \mu_2)) \neq \emptyset \\ \text{then the set } \phi^{-1}(h \cap A_2) \text{ is contained in a hyperplane.} \end{cases}$$

The μ_1-measure of the different closed half-spaces H and H' with common boundary hyperplane h is between 0 and 1. By (6.4) this is also valid for the μ_2-measure of $J^*(H)$ and $J^*(H')$. Since by (6.4)

$$\vartheta_{\mu_2}(J^*(H), J^*(H')) = \vartheta_{\mu_1}(H, H') = 1$$

holds, we obtain from this

$$\mu_2(J^*(H) \cap J^*(H')) = 0 \ \text{ and } \ \mu_2(J^*(H) \cup J^*(H')) = 1$$

and thus we see by (3) that

$$g := J^*(H) \cap J^*(H')$$

is a hyperplane. From

$$p \in h \cap A_1$$

we infer $p \in H$ and $p \in H'$ and thus by (6.5) $\phi(p) \in J^*(H) \cap J^*(H')$; i.e., $\phi(p)$ is contained in the hyperplane g and thus (6.9) is proved. (6.10) can be proved analogously.

Next we show:

$$\text{(6.11)} \qquad \begin{cases} \text{If a hyperplane } h \text{ does not intersect} \\ \qquad\qquad A_1 \cap int(conv(tr(\mathbb{E}^d, \mu_1))) \\ \text{then } \phi(h \cap A_1) \text{ is also contained in a hyperplane.} \end{cases}$$

We argue indirectly. If the set $\phi(h \cap A_1)$ is not contained in a hyperplane then there exists an $\varepsilon > 0$ such that

(6.12) $\mu_2(H) \geq \varepsilon$ holds for any half-space H containing $\phi(h \cap A_1)$.

Since $\phi(h \cap A_1)$ is not contained in a hyperplane it contains $d+1$ affinely independent points $\{x_0, ..., x_d\}$ and there exist neighbourhoods $\{U(x_0), ..., U(x_d)\}$ of these points such that any half-space containing $\{x_0, ..., x_d\}$ also contains one of these neighbourhoods. Since $\{x_0, ..., x_d\}$ is a subset of $tr(\mathbb{E}^d, \mu_1)$ we get that μ_1 does not vanish on any of the neighbourhoods. Thus if we let $\varepsilon = \min(\mu_1(U(x_0)), ..., \mu_1(U(x_d))$ we see that (6.12) holds.

However the μ_1-measure of one of the two half-spaces with boundary hyperplane h is 0. We denote this half-space by H. Let now $\langle H_n \rangle_{n \in \mathbf{N}}$ be a monotonically decreasing sequence of closed half-spaces containing H such that

$$\mu_1(H_n) > 0 \ \text{ and } \ \lim_{n \to \infty} \mu_1(H_n) = 0 \, .$$

Then we see that

$$\phi(h \cap A_1) \subset J^*(H_n) \ \text{ and, by (6.4), } \ \lim_{n \to \infty} \mu_2(J^*(H_n)) = 0 \, ;$$

i.e., there exists an $n \in \mathbf{N}$ such that for the half-space $J^*(H_n)$

$$\phi(h \cap A_1) \subset J^*(H_n) \ \text{ and } \ \mu_2(J^*(H_n)) < \varepsilon$$

holds, in contradiction to (6.12). Thus (6.11) is proved.

Analogously to (6.11) we may prove:

(6.13) $$\begin{cases} \text{If for a half-space } h \\ \qquad h \cap A_2 \cap int(conv(tr(\mathbb{E}^d, \mu_2))) = \emptyset \\ \text{holds then the set } \phi^{-1}(h \cap A_2) \text{ is contained in a hyperplane.} \end{cases}$$

By (6.9) and (6.11) we finally get that subsets of hyperplanes contained in A_1 are mapped by ϕ into subsets of hyperplanes; by (6.10) and (6.13) we obtain that ϕ^{-1} maps also subsets of hyperplanes contained in A_2 into hyperplanes. Thus (6.1) is proved.

Next we show that ϕ maps affinely independent sets to affinely independent sets. In $\mathbb{E}^d$ it is sufficient to consider $d+1$-point sets, since we may extend our set to a $d+1$-point affinely independent set.

We argue indirectly. Let $M = \{x_0, ..., x_d\} \subset A_1$ be an affinely independent set. If $\phi(M)$ were not affinely independent then $\phi(M)$ would be contained in a hyperplane, since it is a $d+1$-point set. But since ϕ^{-1} maps subsets of hyperplanes into subsets of hyperplanes, the set $M = \phi^{-1}(\phi(M))$ would also be contained in a hyperplane. This contradiction shows that ϕ preserves the affine independence of a set.

Analogously we can show that ϕ^{-1} also preserves the affine independence of a set. From this we infer by negation that ϕ also preserves affine dependence and thus preserves the affine dimension (i.e.,the dimension of the affine hull) of a set. Namely the affine hull of a set M is n-dimensional if and only if M contains $n+1$ affinely independent points and any $n+2$-point subset of M is affinely dependent.

The last statement of Theorem 1, i.e., that ϕ is an affinity if $tr(\mathbb{E}^d, \mu_1) = \mathbb{E}^d$ or $tr(\mathbb{E}^d, \mu_2) = \mathbb{E}^d$, can now be concluded from Lemma 2.□

Remark 2

Parts 1 and 2 of the proof show that, under the given hypotheses, there exists no isometry from the space of equivalence classes of halfspaces of $\mathbb{E}^d$ into the space of equivalence classes of convex sets of a lower dimensional space.

Corollary 1

Let μ_1 and μ_2 be finite measures on $\mathbb{E}^d$ with $\mu_1(\mathbb{E}^d) = \mu_2(\mathbb{E}^d)$ such that

$$\lambda << \mu_i << \lambda \text{ for } i \in \{1, 2\}$$

holds, where λ denotes the Lebesgue measure. Then any isometry

$$I : (\mathcal{Q}_{-\emptyset}, \vartheta_{\mu_1}) \to (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_2})$$

is induced by an affinity $l : \mathbb{E}^d \to \mathbb{E}^d$.□

§4 CONSEQUENCES OF THE MAIN THEOREM

Theorem 2

Let $G_1, G_2 \subset \mathbb{E}^d$, where G_2 is open and connected and let μ_1 and μ_2 be measures on G_1 and G_2 respectively, such that

$$\mu_1(G_1) = \mu_2(G_2) =: m < \infty$$

and

$$\lambda\,|_{G_i} << \mu_i << \lambda\,|_{G_i} \quad \text{for } i \in \{1, 2\}\ .$$

Let $\mathcal{D}_1$ fulfill the (parts concerning $\mathcal{D}_1$ of the) hypotheses (4) and (6) and let μ_1 fulfill hypothesis (3) of Theorem 1. Then any isometry

$$I : (\mathcal{D}_1, \vartheta_{\mu_1}) \to (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_2})$$

is induced by a (with respect to μ_1 and μ_2) measure preserving projective mapping $f : \mathbb{E}^d \to P^d$; i.e., there exists a measure preserving projective mapping f such that

$$I([D]) = [f(D)]_2$$

holds if $[D] \in \mathcal{D}_1$.

Proof: As easily seen the hypotheses (1)-(6) of Theorem 1 hold for the measures $\frac{\mu_1}{m}$ and $\frac{\mu_2}{m}$ and thus, since G_2 is open, there exist sets A_1 and A_2 with $A_2 \supseteq G_2$ and a homeomorphism

$$\phi : A_1 \to A_2$$

which leaves the dimension of the affine hull of a set invariant and induces I.

Since ϕ is a measure preserving homeomorphism and $A_2 \supseteq G_2$ holds, the set $\phi^{-1}(G_2)$ is connected, by the theorem on the invariance of domain open and $[\phi^{-1}(G_2)] = [G_1]$. We may therefore assume w.l.o.g. that $\phi(G_1) = G_2$ and G_1 is an open connected set, since this assumption does not change $tr(\mathbb{E}^d, \mu_1)$ or the

μ_1-equivalence classes.

Thus from Theorem 1 and Lemma 1 we infer that ϕ is a projective mapping, inducing I.□

An immediate consequence of Theorem 2 is the following generalization of [Gr2]:

Theorem 3

Let λ be the Lebesgue measure on $\mathbb{E}^d$ and let $G_1, G_2 \subset \mathbb{E}^d$ with G_2 open and connected and $\lambda(G_1) = \lambda(G_2) < \infty$. Let $\mu_1 = \lambda \mid_{G_1}$, $\mu_2 = \lambda \mid_{G_2}$, and let $\mathcal{D}_1$ fulfill the (parts concerning $\mathcal{D}_1$ of the) hypotheses (4) and (6) and let μ_1 fulfill hypothesis (3) of Theorem 1. Then any isometry

$$I : (\mathcal{D}_1, \vartheta_{\mu_1}) \to (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_2})$$

is induced by a (with respect to λ) measure preserving affinity $l : \mathbb{E}^d \to \mathbb{E}^d$; i.e., there exists a measure preserving affinity l such that

$$I([D]) = [l(D)]_2$$

if $[D] \in \mathcal{D}_1$.□

Remark 3

To get a better impression how this result depends on the special geometric properties of Lebesgue measure we give now a proof, which shows that Theorem 3 can be deduced directly from Theorem 1 without using Theorem 2 and thus without using the fundamental theorem of projective geometry.

Proof of Theorem 3: In analogy to the first steps in the proof of Theorem 2 we see that we can assume w.l.o.g. that I is induced by a homeomorphism $\phi : G_1 \to G_2$ which fulfills the assertions of Theorem 1 about ϕ and that G_1 is connected and open

Since ϕ and ϕ^{-1} are continuous and preserve the affine hull of a set, they map line segments onto line segments and thus convex sets onto convex sets. Since ϕ and ϕ^{-1} are homeomorphisms we get:

(i) The image $\phi(C)$ of a compact convex set $C \subset G_1$ is compact and convex and so is the image $\phi^{-1}(C)$ of a compact convex set $C \subset G_2$.

We show first:

(ii) The ϕ-image of a closed d-dimensional simplex $S \subseteq G_1$ is again a closed d-dimensional simplex $\phi(S)$.

Let $S \subseteq G_1$ be a d-dimensional simplex and let $\mathcal{E} = \{p_0, ..., p_d\}$ be the set of its vertices. Then we get, since $\phi(S)$ is by (i) convex,

$$conv(\phi(\mathcal{E})) \subseteq \phi(S) \tag{iii}$$

.

Since $\phi^{-1}(conv(\phi(\mathcal{E})))$ is by (i) convex, and thus

$$conv(\mathcal{E}) \subseteq \phi^{-1}(conv(\phi(\mathcal{E})))$$

holds, we obtain with (iii), that

$$\phi(S) = \phi(conv(\mathcal{E}) \subseteq \phi(\phi^{-1}(conv(\phi(\mathcal{E})))) =$$

$$conv(\phi(\mathcal{E})) \subseteq \phi(S)$$

and thus

$$\phi(S) = conv(\phi(\mathcal{E}))$$

holds. Since ϕ preserves by Theorem 1 the dimension of the affine hull of a set, $\phi(\mathcal{E})$ is affinely independent and thus $\phi(S)$ is in fact a d-dimensional simplex so that (ii) is proved.

Further we show:

(iv) ϕ is a local affinity.

Let $S \subset G_1$ be a d-dimensional simplex, $\mathcal{E} = \{p_0, ..., p_d\}$ the set of its vertices and $x \in int(S)$. We get by (ii) that the d-dimensional simplices

$$S^x_{p_n} = conv(\{x\} \cup (\mathcal{E} \setminus \{p_n\})), \;\; n \in \{0, ..., d\}$$

are mapped by ϕ onto d-dimensional simplices

$$S^{\phi(x)}_{\phi(p_n)} = conv(\phi(\{x\} \cup (\mathcal{E} \setminus \{p_n\}))), \;\; n \in \{0, ..., d\}$$

and, since ϕ is by Theorem 1 measure-preserving that

$$\mu_1(S^x_{p_n}) = \mu_2(S^{\phi(x)}_{\phi(p_n)}) \tag{v}$$

holds. Let $h_{p_n}(x)$ be the mapping which assigns to a point $x \in S$ the distance of $\phi(x)$ to the hyperplane $aff(\phi(\mathcal{E} \setminus \{p_n\}))$. Since the volume of a d-dimensional simplex with given $d-1$-dimensional base is proportional to its height we get by (v) that $h_{p_n}(.)$ is affine in x. Since $h_{p_n}(x)$ ($n \in \{1, ..., d\}$) are the coordinates of a point $\phi(x)$ with respect to a coordinate system with axes orthogonal to $\{aff(\phi(\mathcal{E} \setminus \{p_n\})) \mid n = 1, ..., d\}$ and with origin in p_0, we see that ϕ is an affinity on $int(S)$.

Thus to any point $p \in G_1$ there exists an open neigborhood $V(p)$ on which ϕ is an affinity; i.e. (iv) is proved.

Finally we show that ϕ is also a global affinity.

The set M of all points $p \in G_1$ for which there exists a neighborhood $W(p)$, on which ϕ coincides with a given affinity l is by (iv) open. It is also closed in G_1. Since let $p \in \partial_{G_1} M$ then $V(p)$ and the open set M intersect in an open set. But if two affinities l_1 and l_2 coincide on an open subset of $\mathbb{E}^d$ we have $l_1 = l_2$. Thus M is clopen in G_1 and since G_1 is connected we get $M = G_1$ or $M = \emptyset$. But since ϕ is a local affinity, there exists an affinity l such that $M \neq \emptyset$ and the theorem is proved.□

From Theorem 3 we obtain the following generalization of [Gr1].

Theorem 4

Let λ be the Lebesgue measure on $\mathbb{E}^d$. Let $G_1 \subseteq \mathbb{E}^d$ be an open connected and $G_2 \subseteq \mathbb{E}^d$ an open convex set such that $\lambda(G_1) = \lambda(G_2) = \infty$ holds. Further let $\mu_1 = \lambda \mid_{G_1}$, $\mu_2 = \lambda \mid_{G_2}$ and

$$(\mathcal{R}_{-\emptyset}, \vartheta_{\mu_1}) \subseteq (\mathcal{D}, \vartheta_{\mu_1}) \subseteq (\mathcal{C}, \vartheta_{\mu_1})\,.$$

Then any isometry

$$I : (\mathcal{D}, \vartheta_{\mu_1}) \to (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_2})$$

is induced by a (with respect to the Lebesgue measure) measure preserving affinity $l : \mathbb{E}^d \to \mathbb{E}^d$.

Proof: Analogously to [Gr1] we can prove that I is measure- and inclusion preserving.

To any $p \in G_1$ we choose now an open simplex $S_p \subset G_1$ such that $p \in S_p$ and denote by S'_p the unique open convex set in the equivalence class $I([S_p])$. Now we define

$$\nu_1^p := \mu_1 \mid_{S_p} \quad \text{and} \quad \nu_2^p := \mu_2 \mid_{S'_p}\,.$$

Since I is measure- and inclusion preserving, we have

$$\nu_1^p(S_p) = \nu_2^p(S'_p)\,,$$

and

$$(\mathcal{Q}_{-\emptyset}, \vartheta_{\nu_1}) \subseteq (\mathcal{R}_{-\emptyset}, \vartheta_{\mu_1})$$

and thus I maps $(\mathcal{Q}_{-\emptyset}, \vartheta_{\nu_1})$ into $(\mathcal{C}_{-\emptyset}, \vartheta_{\nu_2})$.

If we denote the restriction of I to $(\mathcal{Q}_{-\emptyset}, \vartheta_{\nu_1})$ by I_p we get by Theorem 3 the existence of a measure preserving affinity $l_p : \mathbb{E}^d \to \mathbb{E}^d$ which induces I_p. Since S_p is a neighbourhood of p and G_1 is open and connected, it is easily proved that $l_p = l_q$ for all $p, q \in G_1$. Thus we can define our affinity l by $l := l_p$ independently of p and see that l induces all isometries I_p. Finally it is easy to establish that I is also induced by l.□

Before we come to the theorems on the hyperbolic plane, we give two examples which contrast the proved theorems.

Example 1

Let f_1, f_2 be strictly convex functions from $\mathbb{E}^{d-1}$ to $\mathbb{R}$ and let $G_1, G_2 \subset \mathbb{E}^d$ be the graphs of f_1 and f_2. Let $\phi : G_1 \to G_2$ be given by

$$\phi(x, f_1(x)) = (x, f_2(x)) \quad \textit{for any } x \in \mathbb{E}^{d-1} \ .$$

Let μ_1 be a measure which vanishes on any hyperplane of $\mathbb{E}^d$ and whose support is G_1, and let μ_2 be the image of μ_1 under ϕ. Then the mapping ϕ induces by

$$I([C]_{\mu_1}) := [\overline{conv(\phi(C \cap G_1))}]_{\mu_2}$$

an isometry

$$I : (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_1}) \mapsto (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_2}) \ ,$$

where $\mathcal{C}_{-\emptyset}$ denotes the family of equivalence classes of closed convex sets on $\mathbb{E}^d$ (excluding the class of the empty set). In general the mapping ϕ is not projective and hypothesis (5) of Theorem 1 is not fulfilled.

(To see that hypothesis (5) is in general not fulfilled consider a non atomic probability measure μ_2 with support $P := \{(x, \frac{x^2+1}{10}) \mid x \in \mathbb{R}\} \subset \mathbb{E}^2$. Let $D := \{(x_1, x_2) \in \mathbb{E}^2 : x_2 - |x_1| \geq 0\}$ and let $D' = \overline{conv(P \setminus D)}$. Then D, D' are closed convex sets with $\mu_2(D), \mu_2(D') > 0$, $[D']_2 = [\mathbb{E}^2]_2 \setminus [D]_2$ and we can w.l.o.g. suppose that $[D]_2, [D']_2 \in (\mathcal{D}_2, \vartheta_{\mu_2})$. But there exists of course no half-space H such that $H \in [D]_2$. Thus Hypothesis (5) is not fulfilled.)□

Definition

Let K be a circle and let $p, q \in K$ be non diametric points. Then we write $[p, q)_K$ for the union of $\{p\}$ with the set of all points contained in the smaller component of $K \setminus \{p, q\}$.

We call a family of concentric spheres discrete and bounded if the set of radii with the relative topology of $\mathbb{R}$ is discrete and bounded.

Example 2

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be discrete and bounded, infinite families of concentric spheres in $\mathbb{E}^d$ containing no sphere of maximal radius. Then there exist probability measures μ_1 and μ_2 in $\mathbb{E}^d$, with support $\overline{\bigcup \mathcal{S}_1}$ and $\overline{\bigcup \mathcal{S}_2}$, respectively, vanishing on any hyperplane and an isometry

$$I : (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_1}) \mapsto (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_2}) \,,$$

if and only if $\bigcup \mathcal{S}_2$ is a homothetic copy of $\bigcup \mathcal{S}_1$.

Proof: Clearly the condition is sufficient.

We also prove that it is necessary. For this we investigate first whether the hypotheses on the family of spheres and measures above imply hypotheses (1)-(3) and (5) of Theorem 1. The fact that (1)-(3) hold is immediate; so only the validity of hypothesis (5) remains to be proved.

I.e., we have to prove that given two closed convex sets $C, C' \subseteq \mathbb{E}^d$ such that

(i) $$[C]_2 = [\mathbb{E}^d] \setminus [C']_2 \,,$$

then there exists a half-space $H \in [C]_2$.

Since this is evident if $[C]_2 = [\emptyset]_2$ or $[C]_2 = [\mathbb{E}^d]_2$, we assume w.l.o.g.

(ii) $$0 < \mu_2(C), \mu_2(C') < 1 \,.$$

Further we assume w.l.o.g. that

$$C, C' \subseteq \overline{\bigcup conv(\mathcal{S}_2)}$$

holds. We show that this implies

(iii) $$C \cap C' = \partial C \cap \partial C' \,.$$

From (iii) and the separation theorem for convex sets we will get that there exists a hyperplane F which separates C and C', and further, since μ_2 vanishes on hyperplanes, that condition (5) of Theorem 1 is fulfilled.

Thus we show (iii).

We argue indirectly. Suppose (iii) were not fulfilled. Since C and (C') are by (ii) convex sets with non-empty interior and we have assumed w.l.o.g. $C, C' \subseteq \overline{\bigcup conv(\mathcal{S}_2)}$, there exists a point $p \in int(C) \cap int(C')$ such that

$$p \in int(conv(S^*)) \subset conv(S^*) \subset int(conv(S^{**})) ,$$

where S^* and S^{**} denote suitable spheres of the family $\mathcal{S}_2$. By (i),(ii) and the convexity of C and C' there exist points

$$q_1 \in C \cap S^{**} \ \text{ and } \ q_2 \in C' \cap S^{**} .$$

Since S^{**} is connected, C, C' are closed and $S^{**} \subset C \cup C'$ holds, this gives the existence of a point

$$q \in S^{**} \cap C \cap C' .$$

Let us put

$$r := S^* \cap \overline{pq}.$$

Then

$$r \in int(C) \cap int(C') \cap S^* ,$$

i.e., r lies in the support $\overline{\bigcup \mathcal{S}_2}$ of μ_2 and $C \cap C'$ is a neighborhood of r. Thus we get $\mu_2(C \cap C') > 0$ which contradicts (i). So (iii) and thus the validity of hypothesis (5) of Theorem 1 has been established.

Since all hypotheses of Theorem 1 are fulfilled now, the existence of an isometry

$$I : (\mathcal{C}, \vartheta_{\mu_1}) \mapsto (\mathcal{C}, \vartheta_{\mu_2})$$

implies that there exists a homeomorphism

$$\phi : \bigcup \mathcal{S}_1 \mapsto \bigcup \mathcal{S}_2$$

which preserves the dimension of the affine hull of a set. (Since the radii of $\mathcal{S}_1$ and $\mathcal{S}_2$ are both bounded, we have by Theorem 1 a homeomorphism $\phi : \overline{\bigcup \mathcal{S}_1} \to \overline{\bigcup \mathcal{S}_2}$. Since the points of local connectedness, i.e. $\bigcup \mathcal{S}_1$ and $\bigcup \mathcal{S}_2$, are preserved by ϕ and ϕ^{-1}, so $\phi(\bigcup \mathcal{S}_1) = \bigcup \mathcal{S}_2$.)

Since the components of $\bigcup \mathcal{S}_1$ and $\bigcup \mathcal{S}_2$ are the elements of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, any element of $\mathcal{S}_1$ is mapped by ϕ homeomorphically onto an element of $\mathcal{S}_2$ and vice versa.

Now we show

(iv) ϕ leaves the ratio of the radii of two arbitrary elements S, S' of $\mathcal{S}_1$ invariant.

Notice that (iv) is clearly a sufficient condition for the homothety of $\bigcup \mathcal{S}_1$ with $\bigcup \mathcal{S}_2$.

Let F be a 2-dimensional affine subspace of $\mathbb{E}^d$ which contains the common midpoint of S and S', and let K, K' be the concentric intersections of S respectively S' with F. Since the dimension of the affine hull of a set remains invariant under ϕ, the sets $\phi(K)$ and $\phi(K')$ - which are thus the intersections of $G := aff(\phi(K))$ with $\phi(S)$ respectively $\phi(S')$ - are also concentric circles.

First we show:

(v) The ratio of the radii of K and K' equals the ratio of the radii of $\phi(K)$ and $\phi(K')$.

We may assume w.l.o.g. $conv(K) \supset K'$. Then we choose a sequence of points $\langle x_n \rangle_{n \in \mathbf{N}}$ in K as follows:
Let $x_0 \in K$ be arbitrary, $x_1 \in K$ different from x_0, on a tangent through x_0 to K' and, if $n \geq 2$ and the points $x_0, ..., x_{n-1}$ have already been chosen, we choose $x_n \in K$ such that $x_{n-1} \neq x_n \neq x_{n-2}$ and that the line through x_n and x_{n-1} is a tangent of K'. So the sequence is definitely determined by x_0 and x_1. It is easily seen that the ratio v of the radii of the circles K and K' is given by

$$\text{(vi)} \qquad v = \cos(\pi \lim_{n \to \infty} \frac{\#\{i \mid 1 \leq i \leq n,\ x_i \in [x_0, x_1)_K\}}{n}) \ .$$

To show that $\langle \phi(x_n) \rangle_{n \in \mathbf{N}}$ is a sequence of points in $\phi(K)$ which follows the same rule as $\langle x_n \rangle_{n \in \mathbf{N}}$, it is by the injectivity of ϕ sufficient to show that the ϕ-images of two points $p, q \in K$ with $g := aff(p, q)$ tangential to K' are two points $\phi(p), \phi(q)$ such that $g_\phi := aff(\phi(p), \phi(q))$ is a tangent of $\phi(K')$. But since g intersects K' in one and only one point r, thus $K \cup K'$ in three points, and since ϕ as well as ϕ^{-1} preserve the affine hull of a set, the line g_ϕ intersects the set $\phi(K) \cup \phi(K')$ also in 3 points and thus $\phi(K')$ in exactly one point; i.e., g_ϕ is indeed a tangent of $\phi(K')$. The ratio v_ϕ of the radii of $\phi(K)$ and $\phi(K')$ is now obtained analogously to (vi) by

$$v_\phi = \cos(\pi \lim_{n \to \infty} \frac{\#\{i \mid 1 \leq i \leq n,\ \phi(x_i) \in [\phi(x_0), \phi(x_1))_{\phi(K)}\}}{n}) \ .$$

Together with

$$\phi([x_0, x_1)_K) = [\phi(x_0), \phi(x_1))_{\phi(K)}$$

and (vi) we get therefore $v = v_\phi$; i.e., the ratio of the radii of K and K' equals the ratio of the radii of $\phi(K)$ and $\phi(K')$. (That $\phi([x_0, x_1)_K) = [\phi(x_0), \phi(x_1))_{\phi(K)}$

follows, since we have by construction

$$x_2 \notin [x_0, x_1)_K \quad \text{and} \quad \phi(x_2) \notin [\phi(x_0), \phi(x_1))_{\phi(K)} \text{ .)}$$

Thus (v) has been proved.

From this we infer by the special choice of F that the ratio of the radii of S and S' is greater than or equal to the ratio of the radii of $\phi(S)$ and $\phi(S')$. On the other hand, if we choose a hyperplane G which contains the common midpoint of $\phi(S)$ and $\phi(S')$ and consider the circles $L := \phi(S) \cap G$ and $L' := \phi(S') \cap G$ as well as $\phi^{-1}(L)$ and $\phi^{-1}(L')$, we conclude as before that the ratio of the radii of S and S' is smaller than or equal to the ratio of the radii of $\phi(S)$ and $\phi(S')$, and thus that equality holds.

I.e., the ratio of the radii of two arbitrary spheres S and S' equals the ratio of the radii of $\phi(S)$ and $\phi(S')$. Thus we have proved assertion (iv) and obtain that $\bigcup \mathcal{S}_2$ is in fact homothetic to $\bigcup \mathcal{S}_1$.□

Proposition 2

Let $S \subset \mathbb{E}^2$ be a closed triangle and $\mathcal{E} = \{p_0, ..., p_2\}$ be the set of its vertices. Let μ be a measure on $\mathbb{E}^2$ which is finite on S and which vanishes on no open subset of S. Then any point $x \in intS$ is uniquely determined by the values

$$x_i := \mu(conv((\mathcal{E} \setminus \{p_i\}) \cup \{x\})), \quad where \;\; i \in \{0, ..., 2\} \,.$$

Proof: Let x and x' be two different points in $int(S)$. Then there exists an $i \in \{0, ..., 2\}$ such that either

$$x \in int_S(conv((\mathcal{E} \setminus \{p_i\}) \cup \{x'\}))$$

or

$$x' \in int_S(conv((\mathcal{E} \setminus \{p_i\}) \cup \{x\}))\,;$$

i.e., in any case $x_i \neq x'_i$ holds.□

Definition

We denote by H^2 the closure of the Caley-Klein model of the hyperbolic plane and by ν the hyperbolic measure on H^2. We call the points of ∂H ideal points and a triangle whose vertices are all in ∂H an ideal triangle. We say further that five points of $\mathbb{E}^2$ are in general position if there exists a unique conic through these points. Note that any five distinct points on a circle are in general position.

Theorem 5

Let $G_1, G_2 \subset H^2$ be closed convex sets with non-empty interiors such that $G_1 \cap \partial H^2$ contains a subset $\mathcal{L} = \{p_1, ..., p_5\}$ of five points and such that $\nu(G_1) = \nu(G_2) < \infty$ holds. Further let

$$\mu_1 = \nu \mid_{G_1} \quad \text{and} \quad \mu_2 = \nu \mid_{G_2}$$

and assume

$$(\mathcal{Q}_{-\emptyset}, \vartheta_{\mu_1}) \subseteq (\mathcal{D}, \vartheta_{\mu_1}) \subseteq (\mathcal{C}, \vartheta_{\mu_1}) \ .$$

Then any isometry

$$I : (\mathcal{D}, \vartheta_{\mu_1}) \to (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_2})$$

is induced by a hyperbolic transformation $\phi : H^2 \to H^2$; i.e., there exists a hyperbolic transformation ϕ such that for all $[D] \in \mathcal{D}$

$$I([D]) = [\phi(D)]_2$$

holds.

Proof: Application of Theorem 2 shows now that there exists a (with respect to μ_1 and μ_2) measure preserving projective mapping $\phi : \mathbb{E}^2 \to P^2$ which induces I. Since H^2 is a projective model of the hyperbolic plane, this is also true for $\phi(H^2)$ and the measure $\phi(\nu)$ is the hyperbolic measure for $\phi(H^2)$.

The set $\phi(\mathcal{L})$ is now contained in $\partial\phi(H^2)$ (the set of ideal points of $\phi(H^2)$). Since ϕ preserves measure (with respect to μ_1 and μ_2) the image $\phi(S)$ of any ideal triangle $S \subset H^2$ is again an ideal triangle in H^2. (This is true since triangles are mapped onto triangles [This is seen for triangles contained in $int(H)$ as it is seen for simplices in the proof of (ii) in Theorem 3, and can then be shown for ideal triangles by a limit argument.], the measure of all ideal triangles in the hyperbolic plane is the same and is maximal on the family of all triangles.) Thus we get also $\phi(\mathcal{L}) \subset \partial H^2$. But by the projectivity of ϕ the set $\phi(\mathcal{L})$ is in general position in $\mathbb{E}^2$. So we get $\phi(H^2) = H^2$ and thus ϕ is a projective isomorphism of H^2 onto itself which preserves the hyperbolic measure ν.

From this we infer that ϕ is a hyperbolic transformation by showing that ϕ coincides on any ideal triangle with some hyperbolic transformation. Let $S \subset H^2$ be a closed, ideal hyperbolic triangle and let $\mathcal{E} = \{p_0, p_1, p_2\}$ be the set of its vertices. Then there exists a hyperbolic transformation

$$\psi_S : H^2 \to H^2$$

such that

$$(*) \qquad \psi_S(p_i) = \phi(p_i) \text{ for } i \in \{0, 1, 2\}$$

holds. Since ϕ and ψ_S are measure preserving we get for any $x \in S$ and any $i \in \{0, 1, 2\}$

$$\phi(x)_i := \nu(conv(\phi(\mathcal{E} \setminus \{p_i\}) \cup \{\phi(x)\}) = \nu(conv((\mathcal{E} \setminus \{p_i\}) \cup \{x\})$$

as well as

$$\psi_S(x)_i := \nu(conv(\psi_S(\mathcal{E} \setminus \{p_i\}) \cup \{\psi_S(x)\}) = \nu(conv((\mathcal{E} \setminus \{p_i\}) \cup \{x\})$$

and thus

$$\phi(x)_i = \psi_S(x)_i \ .$$

By Proposition 2 and $(*)$ we get further $\phi(x) = \psi_S(x)$ and thus that ϕ and ψ_S coincide on S, which shows by the arbitrariness of S that ϕ is the desired transformation.□

Proposition 3

Let $I : (\mathcal{D}_1, \vartheta_{\mu_1}) \to (\mathcal{D}_2, \vartheta_{\mu_2})$ be a measure preserving isometry and let $(\mathcal{D}, \vartheta_{\mu_1})$ be a subspace of $(\mathcal{D}_1, \vartheta_{\mu_1})$ such that for any $[C] \in (\mathcal{D}_1, \vartheta_{\mu_1})$ there exists a countable subspace $(\mathcal{D}_C, \vartheta_{\mu_1})$ of $(\mathcal{D}, \vartheta_{\mu_1})$ such that:

(i) $[D], [D]' \in (\mathcal{D}_C, \vartheta_{\mu_1})$ *implies* $[D] \cap [D'] = [\emptyset]$.

and

(ii) $[C] = \bigcup_{D \in \mathcal{D}_C} [D]$.

Suppose further that there exists a mapping ϕ which induces $I \mid_{\mathcal{D}}: (\mathcal{D}, \vartheta_{\mu_1}) \to (\mathcal{D}_2, \vartheta_{\mu_2})$; i.e., $[\phi(D)]_2 = I([D])$ for all $D \in (\mathcal{D}, \vartheta_{\mu_1})$.
Then ϕ induces also I; i.e., $[\phi(D_1)]_2 = I([D_1])$ for all $D_1 \in (\mathcal{D}_1, \vartheta_{\mu_1})$.

Proof: Since I is by Proposition 1 inclusion preserving, we have

$$\text{(iii)} \qquad I([C]) \supseteq \bigcup_{D \in \mathcal{D}_C} I([D]) \ .$$

Further, since I is a measure preserving isometry, we have for $[D], [D'] \in (\mathcal{D}_C, \vartheta_{\mu_1})$

$$[D] \cap [D'] = [\emptyset] \Leftrightarrow \vartheta_{\mu_1}(D, D') = \mu_1(D) + \mu_1(D') \Leftrightarrow$$

$$\vartheta_{\mu_2}(I([D]), I([D'])) = \mu_2(I([D])) + \mu_2(I([D'])) \Leftrightarrow I([D]) \cap I([D']) = [\emptyset] \ .$$

So we obtain from (i) the "disjointness" of $\{I([D]) \mid D \in \mathcal{D}_C)\}$ and thus obtain by (ii) and the hypothesis that I is measure preserving that

$$\mu_2(I([C])) = \mu_1(C) = \sum_{D \in \mathcal{D}_C} \mu_1(D) = \sum_{D \in \mathcal{D}_C} \mu_2(I[D]) = \mu_2(\bigcup_{D \in \mathcal{D}_C} I([D]))$$

and thus by (iii) that

(iv) $$I([C]) = \bigcup_{D\in\mathcal{D}_C} I([D])\ .$$

Since by our hypothesis on ϕ we have

$$\bigcup_{D\in\mathcal{D}_C} I([D]) = \bigcup_{D\in\mathcal{D}_C} [\phi(D)]_2 = [\bigcup_{D\in\mathcal{D}_C} \phi(D)]_2 = [\phi(\bigcup_{D\in\mathcal{D}_C} D)]_2 = [\phi(C)]_2\ ,$$

we obtain from (iv) that $I([C]) = [\phi(C)]_2$.□

Definition

Given a measure ν on $\mathbb{E}^2$ we denote by $(\mathcal{E}, \vartheta_\nu)$ the space of all equivalence classes of convex polygons in $\mathbb{E}^2$ whose number of vertices is less than or equal to 6. We let $(\mathcal{E}_{-\emptyset}, \vartheta_\nu) = (\mathcal{E}, \vartheta_\nu) \setminus \{[\emptyset]_\nu\}$.

In all the results we obtained so far we made use of the fact that $(\mathcal{H}_{-\emptyset}, \vartheta_{\mu_1}) \subseteq (\mathcal{D}_1, \vartheta_{\mu_1})$. This is contrasted by the following Lemma 3 which uses instead the fact that I is surjective and that $(\mathcal{D}_2, \vartheta_{\mu_2})$ is a closed subspace of $(\mathcal{C}_{-\emptyset}, \vartheta_{\mu_2})$. Further Lemma 3 is essential for the proof of Theorem 6, which concerns isometries in the hyperbolic plane.

Lemma 3 (see also [Gr1])

Let μ_1, μ_2 be two measures on $\mathbb{E}^d$. Suppose that μ_2 vanishes on the boundary ∂C of any convex set $C \subset \mathbb{E}^d$. Let $(\mathcal{D}_1, \vartheta_{\mu_1}) \subseteq (\mathcal{C}_{-\emptyset}, \vartheta_{\mu_1})$ and let $(\mathcal{D}_2, \vartheta_{\mu_2})$ be a closed subspace of $(\mathcal{C}_{-\emptyset}, \vartheta_{\mu_2})$. Further suppose that there exists a sequence $\langle D_n\rangle_{n\in\mathbf{N}}$ of closed convex sets with $[D_n] \in (\mathcal{D}_1, \vartheta_{\mu_1})$ and $\lim_{n\to\infty} \mu_1(D_n) = 0$. Then any surjective isometry

$$I : (\mathcal{D}_1, \vartheta_{\mu_1}) \to (\mathcal{D}_2, \vartheta_{\mu_2})$$

is measure preserving.

Proof: We argue indirectly. If I is not measure preserving, there exists either

(i) a convex set C and a $\varepsilon > 0$ with $\mu_1(C) > \mu_2(I([C])) + \varepsilon$

or

(ii) a convex set C with $\mu_1(C) + \varepsilon < \mu_2(I([C]))$.

Assume (i), i.e., $\mu_1(C) > \mu_2(I([C])) + \varepsilon$, and let D be a convex set in $(\mathcal{D}_1, \vartheta_{\mu_1})$ with $\mu_1(D) < \frac{\varepsilon}{2}$. Then we obtain $\mu_2(I([D])) > \frac{\varepsilon}{2}$, since otherwise we would have

$$\mu_1(C) - \mu_1(D) > \mu_2(I[C]) + \mu_2(I([D]))\ ,$$

which by

$$\mu_1(C) - \mu_1(D) \le \vartheta_{\mu_1}(C, D) =$$
$$= \vartheta_{\mu_2}(I([C]), I([D])) \le \mu_2(I([D])) + \mu_2(I([C]))$$

contradicts the fact that I is an isometry. This proves $\mu_2(I(D)) > \frac{\varepsilon}{2}$ in case (i).

Now we assume (ii), i.e. $\mu_1(C) + \varepsilon < \mu_2(I[(C)])$. Then $\mu_1(D) < \frac{\varepsilon}{2}$ implies again $\mu_2(I([D])) > \frac{\varepsilon}{2}$, since otherwise we would get

$$\mu_1(C) + \mu_1(D) < \mu_2(I([C])) - \mu_2(I([D]))\,,$$

which by

$$\mu_1(C) + \mu_1(D) \ge \vartheta_{\mu_1}(C, D) =$$
$$= \vartheta_{\mu_2}(I([C]), I([D])) \ge \mu_2(I([C])) - \mu_2([D])$$

contradicts again the fact that I is an isometry. This proves $\mu_2(I(D)) > \frac{\varepsilon}{2}$ in case (ii).

Thus we see that in any case there exists a $\delta > 0$ such that for any C with $[C] \in (\mathcal{D}_1, \vartheta_{\mu_1})$

$$\mu_1(C) < \delta \;\Rightarrow\; \mu_2(I([C])) > \delta$$

holds. We assume now w.l.o.g. that $\vartheta_{\mu_1}(D_m, D_n) < \frac{1}{2^N}$ for $m, n > N$ and that $\mu_1(D_n) < \delta \; \forall n > N$. We choose now out of any equivalence class $I([D_n])$ a convex representative D^n. The set D^n possesses then μ_2-measure greater than δ, and for $m, n > N$ we get $\vartheta_{\mu_2}(D^m, D^n) < \frac{1}{2^N}$. The set

$$D := \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} D^n$$

is convex and of finite measure. Since $[D]_2$ is the limit of $\langle [D^n]_2 \rangle_{n \in \mathbf{N}}$ in $(\mathcal{M}, \vartheta_{\mu_2})$, we have $\mu_2(D) \ge \delta$. Since by assumption $\mu_2(\partial D) = 0$, we have $cl(D) \in [D]_2$. Since $(\mathcal{D}_2, \vartheta_{\mu_2})$ is closed in $(\mathcal{C}_{-\emptyset}, \vartheta_{\mu_2})$ and thus $[D]_2 \in (\mathcal{D}_2, \vartheta_{\mu_2}) \subset (C_{-\emptyset}, \vartheta_{\mu_2})$, and since I is by assumption a surjection we obtain

$$I^{-1}([D]_2) = I^{-1}(\lim_{n \to \infty} [D^n]_2) = \lim_{n \to \infty} [D_n]_1 = [\emptyset].$$

This is a contradiction, since $[\emptyset]$ is not an element of $(\mathcal{C}_{-\emptyset}, \vartheta_{\mu_1})$ and thus is not in $(\mathcal{D}_1, \vartheta_{\mu_1})$. This proves our lemma.□

Theorem 6

Let ν be the hyperbolic measure on H^2, let

$$(\mathcal{E}_{-\emptyset}, \vartheta_\nu) \subseteq (\mathcal{D}_1, \vartheta_\nu) \subseteq (\mathcal{C}_{-\emptyset}, \vartheta_\nu)\,,$$

and let

$$(\mathcal{D}_2, \vartheta_\nu) \;\; \textit{be a closed subspace of} \;\; (\mathcal{C}_{-\emptyset}, \vartheta_\nu) \, .$$

Then any surjective isometry

$$I : (\mathcal{D}_1, \vartheta_\nu) \to (\mathcal{D}_2, \vartheta_\nu)$$

is induced by a hyperbolic transformation $\phi : H^2 \to H^2$.

Proof: We denote by S an arbitrary closed (perhaps ideal) triangle. Since $(\mathcal{E}_{-\emptyset}, \vartheta_\nu)$ contains all equivalence classes of triangles, we get by Lemma 3 (with $\mu_1 = \mu_2 := \nu$) that $\nu(S) = \nu(I([S]))$. Further I is by Lemma 3 and Proposition 1 inclusion preserving and thus the restriction I_S of I to the space $(\mathcal{Q}_{-\emptyset}, \vartheta_{\nu|_S})$ is also inclusion preserving. It is also clear that there exists a closed convex subset T of H^2 such that $I([S]) = [T]$.

By our assumption on $(\mathcal{D}_1, \vartheta_\nu)$ we have

$$(\mathcal{Q}_{-\emptyset}, \vartheta_{\nu|_S}) \subseteq (\mathcal{E}_{-\emptyset}, \vartheta_\nu) \subseteq (\mathcal{D}_1, \vartheta_\nu) \, .$$

We thus get by Theorem 2 a projective mapping $\phi_S : \mathbb{E}^d \to P^d$ which induces the isometry I_S. Considering overlapping triangles, we see that ϕ_S is independent of S. We call this mapping ϕ. Let $(\mathcal{D}, \vartheta_\nu)$ be the space of equivalence classes of hexagons E such that there exists a hyperbolic triangle S with $E \subset S$. Then we get by an application of Proposition 3 that I is induced by the measure preserving mapping ϕ.

To get the result it is now sufficient to consider a closed convex set with nonempty interior $G_1 \subset H^2$ which is spanned by a set $\mathcal{L}$ of five points such that $\mathcal{L} \subset \partial H^2$. The ϕ-image of G_1 is then a set G_2 which is closed convex with nonempty interior and contained in H^2. Since ϕ is measure preserving, ϕ induces an isometry I' from $(\mathcal{Q}_{-\emptyset}, \vartheta_{\nu|_{G_1}})$ into $(\mathcal{C}_{-\emptyset}, \vartheta_{\nu|_{G_2}})$. An application of Theorem 5 shows now that ϕ is a hyperbolic transformation inducing I' and thus it is the desired hyperbolic transformation inducing I.$\square$

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