A characterization of parallelepipeds related to weak derivatives

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Abstract

We characterize in this paper parallelepipeds in $\mathbb{R}^m$ within the family of all convex bodies by a property of special measures on its boundary. We show that these measures are related to weak derivatives (in the sense of [5] and [8]) of convex-valued functions. The results can be applied (see [9]) to derive a generalization of a theorem of Lehmann (see [4]) on the comparison of uniform location experiments.

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1. Introduction

The investigation of special convex bodies and especially their characterization is a main topic in classical convex geometry (see [2] chapter 1.11). There exist several characterizations of parallelepipeds, simplexes and ellipsoids within the family of convex bodies (see [3], [1], [7] and [2]). In section 2 of this paper we give a new characterization of parallelepipeds by a property of certain measures on the boundary of convex sets. In section 3 we define measure valued weak derivatives of convex valued mappings. We show that the measures used to characterize parallelepipeds in section 2 are related to the derivatives of convex valued mappings given by shifts of the parallelepiped (convex body) in the directions of its 1-dimensional edges. We note that the results of section 3 can be used to prove a theorem on the comparison of uniform location experiments

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[9] which generalizes Theorem 3.1 of [4].

We denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of positive integers. By $\mathbb{S}^{m-1}$ we denote the unit sphere in $\mathbb{R}^m$, by $\langle ., . \rangle$ we denote the euclidean scalar product on $\mathbb{R}^m$ and by $\|.\|_2$ we denote the norm associated with $\langle ., . \rangle$. By $\overline{V}$ we denote the closure of a set V, by $\mathbf{1}_B$ we denote the indicator function of a set B and by δ_x we denote the Dirac measure at a point x. Given an at most countable set M we denote by δ_M the counting measure on M, i.e. $\delta_M(A) = \sum_{m \in M} \delta_m(A)$. Given two Borel measures μ and ν on $\mathbb{R}^m$ we denote by $\mu * \nu$ the convolution of μ and ν defined on Borel sets $B \subset \mathbb{R}^m$ by $(\mu * \nu)(B) := (\mu \otimes \nu)(\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \text{ such that } x + y \in B\})$. The same definition applies to signed Borel measures on $\mathbb{R}^m$, provided the convolution exists. If ν is absolutely continuous with respect to μ we write $\nu << \mu$ and we denote by $\frac{d\nu}{d\mu}$ the Radon-Nikodým derivative of ν with respect to μ. For signed measures ν, μ, $\nu << \mu$ means that the total variation of ν is absolutely continuous with respect to the total variation of μ. By λ we denote Lebesgue measure on $\mathbb{R}^m$. Given a measure μ and a measurable function f we denote by $f \cdot \mu$ the signed measure defined by $[f \cdot \mu](A) := \int_A f \, d\mu$ (if it exists). We denote by $aff(C)$ the affine hull of a set C. We say that a convex set C is parallel to a convex set D and write $C \| D$ if $aff(C) \subseteq aff(D) + v$ or $aff(D) \subseteq aff(C) + v$ for some $v \in \mathbb{R}^m$. Given a measure μ and a μ-measurable set Y we let $\mu \mid_Y (A) := \mu(A \cap Y)$ denote the restriction of μ to the μ-measurable subsets of Y.

2. The characterization Theorem

We state and prove in this section Theorem 1 which says that a convex body C (a convex compact subset of $\mathbb{R}^m$ with nonempty interior) is a parallelepiped if and only if certain measures on the boundary ∂C of C possess a certain property.

DEFINITION 1 *Given a convex body $C \subset \mathbb{R}^m$ and $x \in \partial C$ we denote by $\nabla_x C$ the set of all hyperplanes of support to C at x. We denote by $\eta : \partial C \to \mathbb{S}^m$ an arbitrary mapping which maps the point $x \in \partial C$ to a vector $\eta(x)$ which is an outward unit normal vector to C in x, i.e., $\langle \eta(x), y - x \rangle \leq 0$ for all $y \in C$ and $\|\eta(x)\|_2 = 1$. Given a convex body C we denote by $o_{\partial C}$ the surface area measure on ∂C.*

REMARK 1 *By an application of Theorem 1.17 of [6] (which is a special case of Rademacher's Theorem (Theorem 1.18 of [6])) there exists a set D with $o_{\partial C}(D) = 0$ such that $\eta(.)$ is uniquely defined on $\partial C \setminus D$.*

DEFINITION 2 *Given a vector $w \in \mathbb{R}^m$ and a convex body $C \subset \mathbb{R}^m$ we denote by ς_w^C the unique signed Borel measure for which $\frac{d\varsigma_w^C}{do_{\partial C}} = \langle \eta(.), w \rangle$. By Remark 1 this definition is independent of the special choice of the function η in Definition 1.*

THEOREM 1 *A convex body $C \subset \mathbb{R}^m$ is a parallelepiped if and only if there exists a linearly independent set $\{w_1, \ldots, w_m\}$ of vectors $w_j \in \mathbb{R}^m$ such that for any $j \in \{1, \ldots, m\}$ there exists an at most countable set $R_j \subset \mathbb{R}$ and an $r_{0j} \in R_j$ such that*

$$[\varsigma_{w_j}^C * \delta_{\{r \cdot w_j | r \in R_j\}}](B) = 0$$

for any Borel measurable set $B \subseteq \partial(C + r_{0j} \cdot w_j)$, i.e. if and only if $\varsigma_{w_j}^C * \delta_{\{r \cdot w_j | r \in R_j\}}$ vanishes on the Borel measurable subsets of $\partial(C + r_{0j} \cdot w_j)$.

REMARK 2 *If the convex body C in Theorem 1 is a parallelepiped then the vectors $w_1, \ldots, w_m$ in Theorem 1 are parallel to the one dimensional edges of C. We note that it follows from Propositions 2 and 3 that the convolution in Theorem 1 is meaningful (it is defined by a convergent series).*

PROOF. If C is a parallelepiped we just have to choose w_i such that $\|w_i\|$ equals the length of the one dimensional edge of C which is parallel to w_i, $R_j = \{-1, 0, 1\}$ and $r_{0j} = 0$ independent of j. Then

$$[\varsigma_{w_j}^C * \delta_{\{r \cdot w_j | r \in R_j\}}](B) = 0 \quad \text{for all Borel measurable sets} \quad B \subseteq \partial(C + r_{0j} \cdot w_j)$$

which proves the only if part. The if part is the consequence of the lemmas 1, 2 and 3 which we state and prove below. □

PROPOSITION 1 *Given a convex body C, a vector $w \neq 0$, a real number β_1 and a point $x \in \partial C + \beta_1 w$. If for arbitrary hyperplanes H*

$$H \| w \;\Rightarrow\; H \notin \nabla_x [C + \beta_1 w]$$

then there exists one and only one $\beta_2 \neq \beta_1$ such that $x \in \partial C + \beta_2 w$.

PROOF. We show that $(x + \mathbb{R}w) \cap int(C + \beta_1 w) \neq \emptyset$. Otherwise there would exist a hyperplane H which separates $x + \mathbb{R}w$ and $C + \beta_1 w$. This hyperplane would fulfil $H \in \nabla_x [C + \beta_1 w]$ and $H \| w$ which contradicts our hypothesis. Thus there exists $y \in (x + \mathbb{R}w) \cap int(C + \beta_1 w)$. By compactness and convexity of C we get that $(x + \mathbb{R}w) \cap (\partial C + \beta_1 w) = (y + \mathbb{R}w) \cap (\partial C + \beta_1 w)$ consists of exactly two points. One of them is x the other one is $x + \gamma w$ for some $\gamma \in \mathbb{R}$. Thus for $\beta_2 \neq \beta_1$ we have $x \in \partial(C + \beta_2 w)$ if and only if $\beta_2 = \beta_1 + \gamma$. □

PROPOSITION 2 *Given a vector $w \neq 0$. Suppose that the intersection of a convex body C and its translate $C + \alpha w$ (with $\alpha \neq 0$) is again a convex body, i.e., $int(C \cap (C + \alpha w)) \neq \emptyset$. Then*

$$o_{\partial C}(\{x \mid x \in \partial C \cap (\partial C + \alpha w) \;\; s.t. \;\; [H \| w \Rightarrow H \notin \nabla_x C]\}) = 0 \tag{1}$$

and

$$o_{\partial C + \alpha w}(\{x \mid x \in \partial C \cap (\partial C + \alpha w) \;\; s.t. \;\; [H \| w \Rightarrow H \notin \nabla_x C]\}) = 0 \tag{2}$$

PROOF. We prove only (1), since the proof of (2) is completely analogous. Let

$$X = \{x \mid x \in \partial C \cap (\partial C + \alpha w) \text{ s.t. } [H \| w \Rightarrow H \notin \nabla_x C]\}.$$

We have to show that $o_{\partial C}(X) = 0$. We show first that $x \in X$ implies that

$$(x + \mathbb{R}w) \cap X = x. \tag{3}$$

We prove (3) indirectly: Suppose that $\exists y \neq x$ such that $y \in (x + \mathbb{R}w) \cap X$. Then $y - x = \gamma w$ with $\gamma \neq 0$ and we obtain from the definition of X that

$$y \in \partial C + 0,\ y \in \partial C + \alpha w,\ y \in \partial C + \gamma w,\ y \in \partial C + (\alpha + \gamma) w\ .$$

Since $\alpha \neq 0$ and $\gamma \neq 0$ three of the four numbers $0, \alpha, \gamma, \gamma + \alpha$ must be pairwise different which contradicts together with the fact that $y \in X$ Proposition 1. So (3) has been proved.

Let $Z := \{z \in \mathbb{R}^m \mid \inf_{\{x \mid z - x \in \mathbb{R}w\}} \|x\|_2 = 1\}$ and let $S := \{z \in Z \mid \|z\|_2 = 1\}$. We suppose without loss of generality that $0 \in int(C \cap (C + \alpha w))$. Then $X \cap \mathbb{R}w = \emptyset$ and thus $X \setminus \mathbb{R}w = X$. We denote by $pr : X \to Z$ the projection of X along the rays from the origin to Z. By the fact that $X \setminus \mathbb{R}w = X$ the mapping pr is well defined. We identify $Z = S + \mathbb{R}w$ with $S \times \mathbb{R}$ and denote by $\mu := o_S \otimes \lambda_1$ the measure on Z which is the product of the surface area measure o_S on S and the Lebesgue measure λ_1 on $\mathbb{R}$. Note that by (3) we get for $x, \tilde{x} \in X$ that

$$\begin{array}{l} x \neq \tilde{x} \text{ implies that for } (s,r), (\tilde{s}, \tilde{r}) \in S \times \mathbb{R} \text{ with} \\ (s,r) = pr(x) \text{ and } (\tilde{s}, \tilde{r}) = pr(\tilde{x}) \text{ we have } s \neq \tilde{s}. \end{array} \tag{4}$$

Then $\mu(pr(X)) = \int_S \int_{\mathbb{R}} \mathbf{1}_{pr(X)}\ d\lambda_1\ do_S = 0$ by Fubini's theorem and (4). Since $o_{\partial C}$ is absolutely continuous with respect to $\mu \circ pr$ we obtain that $o_{\partial C}(X) = 0$ which proves the proposition. □

We state now two further propositions whose easy proofs are left to the reader.

PROPOSITION 3 *Given a convex body C and a vector $w \in \mathbb{R}^m \setminus \{0\}$, then there exist exactly two numbers α_1 and $\alpha_2 \in \mathbb{R}$ such that $C \cap (C + \alpha_i w) \neq \emptyset$ and $int(C \cap (C + \alpha_i w)) = \emptyset$. Further there exists a hyperplane F such that $F \| C \cap (C + \alpha_i w)$.*

PROPOSITION 4 *Given a convex body C and a vector $w \neq 0$ then*

$$o_{\partial C}\{x \in \partial C \ \ s.t.\ \ [H \| w \Rightarrow H \notin \nabla_x C]\} > 0$$

LEMMA 1 *Given an at most countable set $R \subset \mathbb{R}$, an element $\alpha_0 \in R$, a convex body C, and a vector $w \in \mathbb{R}^m \setminus \{0\}$. Suppose that*

$$[\varsigma_w^C * \delta_{\{r \cdot w \mid r \in R\}}](B) = 0 \ \ \textit{for all Borel measurable sets } B \subseteq \partial(C + \alpha_0 w)\ . \tag{5}$$

Then the following conclusion holds:
There exists a hyperplane $F \not\parallel w$ *such that*

$$x \in \partial C \Rightarrow [\exists H \in \nabla_x C \;\; s.t. \;\; H \| w \;\; or \;\; \exists H \in \nabla_x C \;\; s.t. \;\; H \| F]$$

PROOF. We suppose without loss of generality that $\alpha_0 = 0$. Let

$$Y = \{x \in \partial C \;\; \text{s.t.} \;\; [H \| w \Rightarrow H \notin \nabla_x C]\}.$$

Then $o_{\partial C}(Y) > 0$ by Proposition 4. Since there exists a choice of $\frac{d\varsigma_w^C}{do_{\partial C}}(y)$ (which is $o_{\partial C}$-a.e. determined) such that $0 \neq \frac{d\varsigma_w^C}{do_{\partial C}}(y)$ for all $y \in Y$ we get that

$$\varsigma_w^C|_Y \neq 0. \tag{6}$$

By Proposition 2 we obtain that if $C \cap (C + \alpha w)$ is a convex body and $0 \neq \alpha$ then

$$\begin{aligned} o_{\partial(C+\alpha w)}(Y) = o_{\partial(C+\alpha w)}(Y \cap \partial(C + \alpha w)) = \\ o_{\partial(C+\alpha w)}(\{x \mid x \in \partial C \cap (\partial C + \alpha w) \;\; \text{s.t.} \;\; [H \| w \Rightarrow H \notin \nabla_x C]\}) = 0 \,. \end{aligned} \tag{7}$$

Since $\varsigma_w^C * \delta_{\alpha w} = \varsigma_w^{C+\alpha w} << o_{\partial(C+\alpha w)}$ we obtain from (7) that $\varsigma_w^C * \delta_{\alpha w}|_Y = 0$ if $C \cap (C + \alpha w)$ is a convex body and $0 \neq \alpha$. Of course $\varsigma_w^C * \delta_{\alpha w}|_Y = 0$ holds also if $C \cap (C + \alpha w) = \emptyset$. Thus we obtain for $\alpha \neq 0$ that

$$\varsigma_w^C * \delta_{\alpha w}|_Y = 0 \;\; \text{if} \;\; C \cap (C + \alpha w) = \emptyset \;\; \text{or} \;\; int(C \cap (C + \alpha w)) \neq \emptyset \,. \tag{8}$$

Since R is at most countable we obtain from (5), (6) and since we supposed $\alpha_0 = 0$ that

$$\sum_{\alpha \in R \setminus \{0\}} \varsigma_w^{C+\alpha w}|_Y = \sum_{\alpha \in R \setminus \{0\}} \varsigma_w^C * \delta_{\alpha w}|_Y = \varsigma_w^C * \delta_{\{r \cdot w \mid r \in R \setminus \{0\}\}}|_Y = -\varsigma_w^C|_Y \neq 0 \,. \tag{9}$$

By Proposition 3 we obtain that there exist exactly two values $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $C \cap (C + \alpha_i w) \neq \emptyset$ and $int(C \cap (C + \alpha_i w)) = \emptyset$ with $i \in \{1, 2\}$. This implies together with (8) and (9) that

$$\sum_{i \in \{1,2\}} \varsigma_w^{C+\alpha_i w} \;|_Y = \sum_{i \in \{1,2\}} \varsigma_w^C * \delta_{\alpha_i w}|_Y >> -\varsigma_w^C \;|_Y \;\; . \tag{10}$$

Since $o_{\partial C} \;|_Y << \; -\varsigma_w^C \;|_Y$ we obtain from (10) that $o_{\partial C}$ almost all points of Y must be contained in the union of $C \cap (C + \alpha_1 w)$ and $C \cap (C + \alpha_2 w)$. Further the sets $C \cap (C + \alpha_1 w)$ and $C \cap (C + \alpha_2 w)$ are by Proposition 3 parallel to one hyperplane F. The remaining null set $A \subset Y$ with respect to o_C cannot contain an open subset of ∂C. Thus $y \in A$ must be an element of $\overline{C \cap (C + \alpha_1 w)}$, $\overline{C \cap (C + \alpha_2 w)}$, or $\overline{\{x \in \partial C \mid \exists H \in \nabla_x C \;\; \text{s.t.} \;\; H \| w\}}$. In the first two cases there exists $H \in \nabla_y C$ s.t. $H \| F$ and in the third case there exists $H \in \nabla_y C$ s.t. $H \| w$. This concludes the proof. $\square$.

LEMMA 2 *Given a convex body C which fulfils the conclusion of Lemma 1. Then $C = C_w + [0, \gamma] \cdot w$ for some real number $\gamma > 0$ and some compact convex set $C_w \not\parallel w$ with $dim(C_w) = m - 1$.*

PROOF. Of course there exist exactly two hyperplanes G^1 and G^2 parallel to F which support C.

Let $y \in int(C)$. Then there exist reals $\alpha_y < \beta_y$ such that $y + \alpha_y w$, $y + \beta_y w \in \partial C$. Since $y + \alpha_y w$ and $y + \beta_y w$ cannot possess a hyperplane of support which is parallel w they must by hypotheses possess a hyperplane of support parallel to F. So we get, possibly by an interchange of G^1 and G^2, that

$$y + \alpha_y w \in G^1 \cap C \quad \text{and} \quad y + \beta_y w \in G^2 \cap C. \tag{11}$$

By (11) it is clear that $\gamma = \beta_y - \alpha_y$ is independent of y and $G^2 = G^1 + \gamma w$. Let $D^1 = G^1 \cap C$ and let $D^2 = G^2 \cap C$. Note that

$$\begin{aligned} D^1 &= G^1 \cap C = \overline{\{y + \alpha_y w \mid y \in int(C)\}} \\ D^2 &= G^2 \cap C = \overline{\{y + \beta_y w \mid y \in int(C)\}}. \end{aligned} \tag{12}$$

Thus D^1 and D^2 are compact convex sets of dimension $m - 1$ with

$$D^2 = D^1 + \gamma w. \tag{13}$$

So we get by (11), (13), (12) and the compactness of C that

$$int(C) \subset conv(D^1 \cup D^2) = D^1 + [0, \gamma]w \subseteq \overline{C} = C. \tag{14}$$

Since $D^1 + [0, \gamma]w$ is compact we get from (14) that $D^1 + [0, \gamma]w = C$. This completes the proof if we let $C_w = D^1$. □

LEMMA 3 *Let $C = C_i + [0, \gamma_i] \cdot w_i$ for $w_1, \ldots, w_m$ linearly independent vectors in $\mathbb{R}^m$ and $C_i \not\parallel w_i$ convex compact sets with $dim(C_i) = m - 1$. Then C is a parallelepiped with 1-dimensional edges parallel with w_i.*

PROOF. Note that $\forall i \in \{1, \cdots, m\}$

$$x \in \partial C \Rightarrow (x \in C_i \;\text{ or }\; x \in C_i + \gamma_i \;\text{ or }\; [H \in \nabla_x(C) \;\Rightarrow\; H \| w_i]). \tag{15}$$

From (15) we derive

$$x \in \partial C \Rightarrow \exists i \;\text{ s.t. }\; [x \in C_i \;\text{ or }\; x \in C_i + \gamma_i]. \tag{16}$$

Indirectly: Otherwise by (15) $H \in \nabla_x(C)$ fulfills $H \| w_i$ $\forall i \in \{1, \ldots, m\}$. Since $\{w_i \mid i = 1, \ldots, m\}$ spans by hypotheses $\mathbb{R}^m$ no hyperplane can be parallel with all w_i. Thus we obtained a contradiction and (16) is proved.

Evidently relint(C_i), relint$(C_i + \gamma_i)$, $i \in \{1, ..., m\}$ (with relint(.) denoting the relative interior of a set) belong to ∂C and are disjoint. Hence by (15) each aff(C_i) is parallel to the linear subspace H_i spanned by $\{w_j \text{ s.t. } j \neq i\}$

Let pr_i be the projection along H_i onto $\mathbb{R}w_i$, i.e., let $pr_i(x) = y \in \mathbb{R}w_i$ s.t. $x - y \in H_i$. Then $C \subseteq \bigcap_{n=1}^{m} pr_i^{-1}(pr_i(C)) =: D$ and D is a parallelepiped which can for all $i = 1, \ldots, m$ be written as $D = D_i + [0, \gamma_i]w_i$ with $D_i \supset C_i$.

So the lemma is proved if we show that $C = D$. We proceed indirectly: Since C is a convex body $C \neq D$ implies that there exists $x \in \partial C$ with $x \notin \partial D$. But by (16) there exists an $i \in \{1, \ldots m\}$ such that $x \in C_i \subseteq D_i \subset \partial D$ or $x \in C_i + \gamma_i w_i \subseteq D_i + \gamma_i w_i \subset \partial D$ which contradicts the fact that $x \notin \partial D$ and thus completes the proof of the lemma.□

3. Weak differentiation and a second Version of the Characterization Theorem

DEFINITION 3 *(See also the introduction of [8] or consult [5] Section 3.2.2.) We denote by $\mathcal{C}_c$ the space of continuous real valued functions with compact support on $\mathbb{R}^m$. We remark that a signed Borel measure μ on $\mathbb{R}^m$ of finite total variation is determined by the integrals $\int \phi \, d\mu$ with $\phi \in \mathcal{C}_c$. Thus we can define the derivative of a set valued function as follows: Let $D : [0, \infty) \to \mathcal{P}(\mathbb{R}^m)$. If there exists a signed Borel measure μ on $\mathbb{R}^m$ of finite total variation such that for any $\phi \in \mathcal{C}_c$*

$$\int \phi \, d\mu = \lim_{h \downarrow 0} \int \phi \, \frac{\mathbf{1}_{D(h)} - \mathbf{1}_{D(0)}}{h} \, d\lambda$$

then we say that μ is the derivative from the right of $D(.)$ at 0. (We also say that μ is the weak limit $\lim_{h \downarrow 0} \frac{\mathbf{1}_{D(h)} - \mathbf{1}_{D(0)}}{h} \cdot \lambda$.)

REMARK 3 *The measure ς_w^C can also be described as follows: Let H be the linear $(m-1)$-dimensional subspace perpendicular to w. Let λ_H denote the $m-1$-dimensional Lebesgue measure on H. Let*

$$Y^+ := \{x \in \partial C \mid \langle \eta(x), w \rangle > 0\}$$

and

$$Y^- := \{x \in \partial C \mid \langle \eta(x), w \rangle < 0\}$$

Let $pr_+ : Y^+ \to H$ respectively $pr_- : Y^- \to H$ be the orthogonal projections onto H. Let $P \subset H$ be the orthogonal projection of C onto H. Then $relint(P) \subseteq pr_+(Y^+) \subseteq P$, $relint(P) \subseteq pr_-(Y^-) \subseteq P$ and we have for any $\phi \in \mathcal{C}_c$ that

$$\int_{x \in P} \|w\|_2 \cdot \phi(pr_+^{-1}(x)) d\lambda_H(x) - \int_{x \in P} \|w\|_2 \cdot \phi(pr_-^{-1}(x)) d\lambda_H(x) = \int \phi \, d\varsigma_w^C$$

These facts can be established easily for polytopes or convex bodies with differentiable boundary and then extend without difficulties to arbitrary convex bodies.

We calculate now the derivative from the right at 0 of the special set valued function $D(h) = C + h$, for a convex body C. (Compare with [8] Example 1.)

PROPOSITION 5 *Let C be a convex body, and let $w \in \mathbb{R}^m \setminus \{0\}$. Then the derivative from the right of $h \mapsto (C + h \cdot w)$ equals ς_w^C, i.e. for each $\phi \in \mathcal{C}_c$ we have*

$$\int \phi \, d\varsigma_w^C = \lim_{h \downarrow 0} \int \phi \, \frac{\mathbf{1}_{(C+h\cdot w)} - \mathbf{1}_C}{h} \, d\lambda \, .$$

PROOF. We use the notation of Remark 3. Further we denote by λ_1 the Lebesgue measure on $\mathbb{R}$. Then we have for Borel measurable sets $\tilde{H} \subset H$ and $R \subset \mathbb{R}$ that

$$\lambda(\tilde{H} + \frac{w}{\|w\|} R) = \lambda_1 \otimes \lambda_H (R \times \tilde{H}) \tag{17}$$

We obtain that

$$\lim_{h \downarrow 0} \int_{\mathbb{R}^m} \phi \, \frac{\mathbf{1}_{(C+h\cdot w)} - \mathbf{1}_C}{h} \, d\lambda =$$

$$\lim_{h \downarrow 0} \int_{\xi, x \in \mathbb{R} \times H} \phi(x + \xi \frac{w}{\|w\|_2}) \, \frac{\mathbf{1}_{(C+h\cdot w)} - \mathbf{1}_C}{h} (x + \xi \frac{w}{\|w\|_2}) \, d[\lambda_1 \otimes \lambda_H](\xi, x) =$$

$$\lim_{h \downarrow 0} \int_{x \in H} \int_{\xi \in \mathbb{R}} \phi(x + \xi \frac{w}{\|w\|_2}) \, \frac{\mathbf{1}_{(C+h\cdot w)} - \mathbf{1}_C}{h} (x + \xi \frac{w}{\|w\|_2}) \, d\lambda_1(\xi) \, d\lambda_H(x) =$$

$$\int_{x \in P} \left[\lim_{h \downarrow 0} \tfrac{1}{h} \int_{\xi \in [0, h \cdot \|w\|_2]} \phi(pr_+^{-1}(x) + \xi \tfrac{w}{\|w\|_2}) \, d\lambda_1(\xi) \right] d\lambda_H(x) -$$

$$- \int_{x \in P} \left[\lim_{h \downarrow 0} \tfrac{1}{h} \int_{\xi \in [0, h \cdot \|w\|_2]} \phi(pr_-^{-1}(x) + \xi \tfrac{w}{\|w\|_2}) \, d\lambda_1(\xi) \right] d\lambda_H(x) =$$

$$\int_{x \in P} \|w\|_2 \cdot \phi(pr_+^{-1}(x)) d\lambda_H(x) - \int_{x \in P} \|w\|_2 \cdot \phi(pr_-^{-1}(x)) d\lambda_H(x) = \int \phi \, d\varsigma_w^C$$

The first equality sign holds by (17), the second by Fubini's Theorem, the fourth by the main theorem of calculus and the last equality sign follows by Remark 3. □

If we denote by $\mathbf{U}(C + h \cdot w)$ the uniform probability distribution on $C + h \cdot w$ and note that for $h \neq 0$

$$\int \phi \, \frac{\mathbf{1}_{(C+h\cdot w)} - \mathbf{1}_C}{h} \, d\lambda = \lambda(C) \cdot \int \phi \, d \frac{\mathbf{U}(C + h \cdot w) - \mathbf{U}(C)}{h}$$

then we can reformulate Proposition 5 as follows:

COROLLARY 1 *Let C be a convex body. Then the weak limit $\lambda(C) \cdot \lim_{h \to 0} \frac{\mathbf{U}(C+h\cdot w) - \mathbf{U}(C)}{h}$ equals ς_w^C, i.e.,*

$$\lambda(C) \cdot \lim_{h \to 0} \int \phi \, d \frac{\mathbf{U}(C + h \cdot w) - \mathbf{U}(C)}{h} = \int \phi \, d\varsigma_w^C \quad \textit{for all } \phi \in \mathcal{C}_c. \tag{18}$$

With Corollary 1 and the fact that $\lambda(C) \neq 0$ we can restate Theorem 1 and a part of the remark following Theorem 1 as follows:

THEOREM 2 *A convex body $C \subset \mathbb{R}^m$ is a parallelepiped if and only if there exists a linearly independent set $\{w_1, \ldots, w_m\}$ of vectors $w_j \in \mathbb{R}^m$ such that for any $j \in \{1, \ldots, m\}$ there exists an at most countable set $R_j \subset \mathbb{R}$ and an $r_{0j} \in R_j$ such that*

$$[\Big(\lim_{h \to 0} \frac{\mathbf{U}(C + h \cdot w_j) - \mathbf{U}(C)}{h} \Big) * \delta_{\{r \cdot w_j | r \in R_j\}}](B) = 0$$

for any Borel measurable set $B \subseteq \partial C + r_{0j} \cdot w_j$ (with lim *denoting the weak limit). If the convex body C is a parallelepiped then the vectors $w_1, \ldots, w_m$ are parallel to the one dimensional edges of C.*

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